An efficient semi-positive definite estimator of realized covariance matrix with asynchronous and noisy high frequency data

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Abstract

The wide availability of high-frequency data stimulates the study of estimating integrated covariance matrix in past several years. However the realized covariance estimators developed cannot ensure semi-positive definition and optimal efficiency simultaneously, due to the existence of asynchrony and microstructure noise of the observed data. In this paper, we provide an approach to construct an efficient semi-positive definite (SPD) integrated covariance matrix estimator. A new synchronizing technique named high frequency filtration is proposed that learns from the dependence information of the asynchronous data and iteratively interpolates synchronous data with large sample size \(n\). Together with a new developed correction approach, we are able to obtain an efficient SPD covariance matrix estimator at the optimal convergence rate \(O_p(n^{-1/4})\). We show that our SPD estimator is consistent and has the same limiting distribution as the efficient estimator. Real data oriented simulation experiments are conducted to demonstrate its finite sample performance.

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1 Introduction

Covariance plays an important role in financial study. It is not only a measure of uncertainty about future returns but also an important input parameter in derivative pricing, hedging, portfolio selection and many others. Nevertheless, covariance is not directly observable. Estimate covariance has been of great interest to both academics and industry alike. With a continuously increasing availability of high frequency (HF) data, the study of estimating integrated covariance matrix has been ignited in past several years. Given the observed ticks or transaction data recorded at high sampling frequency, realized variance/covariance has been considered as a desirable ex-post estimator. For a single univariate data, realized variance that is defined as the sum of squared HF returns shows nice property and improved accuracy compared to the concurrent variance estimators based on a coarser e.g. daily sampling frequency on GARCH or others, see e.g. French, Schwert and Stambaugh (1987), Andersen and Bollerslev (1998).

The realized variance is eventually a biased estimator, as the HF data are usually contaminated with microstructure noise. In order to obtain consistent estimators, many methods have been developed, McAleer and Medeiros (2008). For example, the two scaled realized variance takes average over the estimators from different sub-samples to eliminate the impact of microstructure noise (Zhang, Mykland and Aït-Sahalia, 2005); the realized kernel employs an autocorrelation correction procedure to get consistent estimator (Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008); the pre-averaging approach averages out noise directly over the price process and then
computes realized variance (Jacod, Li, Mykland, Podolskij and Vetter, 2009).

Prompted by the growing theoretical and empirical interests, their multivariate versions, the respective realized covariance estimators, have been proposed, see Zhang (2011), Wang and Zou (2010) for the multivariate two scaled estimator, Barndorff-Nielsen, Hansen, Lunde and Shephard (2011) for the multivariate realized kernel, and Christensen, Kinnebrock and Podolskij (2010) for the multivariate pre-averaging estimator. While the first two covariance estimators have relatively low convergence rate at $O_p(n^{-1/6})$ and $O_p(n^{-1/5})$ respectively, the pre-averaging covariance estimator reaches the optimal rate at $O_p(n^{-1/4})$, where $n$ denotes the sample size. However the existing estimators cannot ensure semi-positive definition and optimal efficiency simultaneously. The bias-correction approaches may destroy the semi-positive definition of covariance estimator. To obtain e.g. semi-positive definite (SPD) pre-averaging estimator, the convergence rate reduces to $O_p(n^{-1/5})$, see Christensen et al. (2010).

In practice, HF data are not only noisy but also asynchronous. How to synchronize real data is another fundamental yet often underestimated challenge. For the noise-free model, Hayashi and Yoshida (2005) proposed a procedure for the covariance estimator which has the advantage that it does not throw away any observations. For the noisy model, the conventional synchronizing techniques such as the previous tick technique (see Wasserfallen and Zimmermann, 1985; Dacorogna, Gençay, Müller, Olsen and Pictet, 2001) and refresh time technique (Barndorff-Nielsen et al., 2008) are imperfect. Among others, the previous tick may distort the dynamics of price processes while introducing artificial observations in the synchronized series. The refresh time, on the other hand, discards much information when one or more component data are sampled at low frequency. The pre-averaging estimator, though with fast convergence, adopts the refresh time synchronizing technique, leading to small sam-
ple size and low efficiency. To retain high frequency or large value of $n$, pairwise or blockwise estimators have been proposed, where each element or block of covariance matrix is separately estimated in order to use as much information as possible, see Hautsch, Kyj and Oomen (2012). However, either the semi-positive definition is not guaranteed or the asymptotic properties are unclear in these estimators.

It is interesting to note that none of the synchronizing techniques employs the dependence information of the observed data, though it can be consistently (but possibly inefficiently) estimated with e.g. synchronized low frequency data. The preliminary estimator is indeed informative and can be used to filter the missing values in the asynchronous data, while keeping the dependence structure. We propose a new synchronizing technique that iteratively filters the missing values of the asynchronous data at high sampling frequency. The algorithm eventually minimizes the distance of the available asynchronous data and an underlying synchronous filtered series learning from the dependence information. We name it High Frequency Filtering (HFF) technique. Although the idea of filtering incomplete data is not new, see e.g. Kalman filter under Gaussian distributional assumption (e.g. Dempster, Laird and Rubin, 1977; Rubin and Little, 2002), the HFF synchronizing technique is distributional free and only assumes smoothness of the filters.

Moreover, we develop a new correction approach that takes absolute value of the eigenvalues. Together with the HFF technique, we provide an efficient semi-positive definite (SPD) integrated covariance matrix estimator. The new estimator has the optimal convergence rate $O_p \left( n^{-1/4} \right)$ with large sample size $n$. We show that our new estimator is consistent and has the same limiting distribution as the efficient estimator. Although we demonstrate the new methods with the pre-averaging realized covariance estimator, they are general and applicable to any other realized covariance estimator.
The remaining of this paper is organized as follows. Section 2 introduces the model setting and notations used in our study. We will also present some existing realized covariance estimators in literature. Section 3 presents the efficient SPD estimator. Section 3.1 shows how to use the HFF technique to obtain synchronous but noisy data. Section 3.2 details the correction approach and the asymptotic results of the efficient SPD estimator. Section 4 conducts real data oriented simulation experiments that illustrates the finite sample performance of the proposed efficient SPD estimator in terms of estimation accuracy and feature interpretation. Section 5 concludes.

2 Notations and realized covariance estimators

We will introduce the model setting and notations used in our study. Three integrated volatility matrix estimators are presented, which will be used in numerical study for comparison.

2.1 The setting

Consider $p$ assets over a time interval $[0, 1]$. Let $X_t$ be the log price at time $t \in [0, 1]$ and assume it follows a continuous time diffusion model,

$$dX_t = \mu_t dt + \sigma_t^T dB_t, \quad t \in [0, 1],$$

(2.1)

where $\mu_t = (\mu_{1t}, \ldots, \mu_{pt})^{*}$ is the drift vector, $B_t = (B_{1t}, \ldots, B_{pt})^{*}$ is a standard $p$-dimensional Brownian motion and $\sigma_t$ is a $p \times p$ matrix. The quadratic variation of
\( X_t \) is given by:

\[
\left[ X_t, X_t \right] = \int_0^t \Sigma_u du = \int_0^t \sigma_u^T \sigma_u du, \quad t \in [0, 1].
\] (2.2)

The integrated volatility matrix, denoted by \( \Sigma \), is defined as:

\[
\Sigma \equiv \int_0^1 \Sigma_u du = \int_0^1 \sigma_u^T \sigma_u du.
\] (2.3)

Our goal is to estimate the integrated volatility matrix \( \Sigma \) given the asynchronous and noisy HF data.

### 2.2 Notations

Suppose the efficient log prices are regularly spaced at \( t_j = j/n \), \( j = 0, \ldots, n \) where \( n \) refers to be the sample size at high sampling frequency. Denote \( X_{t_j} = (X_{1,t_j}, \ldots, X_{p,t_j})^* \) to be the efficient log prices of the assets at time \( t_j \). The efficient log prices are synchronous and noise-free, but unobservable in practice.

First, the observed log prices are contaminated by microstructure noise. We assume the existence of additive noise in the price process. That is for all \( i = 1, \ldots, p \) and \( j = 0, \ldots, n \), we have:

\[
Y_{t_j} = X_{t_j} + \epsilon_{t_j}
\]

where \( \epsilon_{t_j} \) represents the microstructure noise which is assumed to be independent and identically distributed (IID) with zero mean and finite variation \( E(\epsilon_{t_j} \epsilon_{t_j}^*) = \Psi \), where \( \Psi \) is diagonal matrix with variation of microstructure noise \( \eta_i^2 \), \( i = 1, \ldots, p \), on the diagonal and symbol * represents the Hermitian transpose. Define the synchronous but noisy log returns as \( R_{i,t_j} = Y_{i,t_j} - Y_{i,t_j-1} \), \( j = 1, \ldots, n \). It can be shown that the
lag-1 autocorrelation of the contaminated log returns is:

\[
\frac{\text{Cov}(R_{i,t-j-1}, R_{i,t})}{\sqrt{\text{Var}(R_{i,t-j-1})\text{Var}(R_{i,t})}} = \frac{-\eta_i^2}{\sqrt{\left(\frac{1}{n} E \int_{t_{j-2}}^{t_{j-1}} \Sigma_{ii,u} du + 2\eta_i^2\right) \left(\frac{1}{n} E \int_{t_{j-1}}^{t_j} \Sigma_{ii,u} du + 2\eta_i^2\right)}} \\
\approx -0.5, \quad j = 2, \ldots, n \tag{2.4}
\]

where \(\Sigma_{ii,u}\) denotes the \((i, i)\)-component of \(\Sigma_u\) of (2.2).

Second, since the observed log prices are irregularly spaced, we define an information set \(\mathcal{F}\) as:

\[\mathcal{F} = \{t_{ij}|Y_{i,t_{ij}} \text{ is available at } t_{ij}, \ i = 1, \ldots, p, \ j = 0, \ldots, n\},\]

which \(t_{i,j}\) represents the time points \(t_j\) only when the log price \(Y_{i,t_{ij}}\) of the \(i\)-th asset is observed at time \(t_j\). Therefore, for any asset \(i\), the set \(\{t_{ij} I(t_{ij} \in \mathcal{F})\}_{j=0}^n\) is an arbitrary subset of \((t_0, t_1, \ldots, t_n)\). Denote \((s_{i1}, \ldots, s_{in})\) to be the set \(\{t_{ij} I(t_{ij} \in \mathcal{F})\}_{j=0}^n\).

Then \((Y_{i,s_{i1}}, \ldots, Y_{i,s_{in}})\) is the observed asynchronous log price of the \(i\)-th asset. Typically \(s_{im} \neq s_{km}\) for \(i \neq k\). If \(t_{ij} \in \mathcal{F}\), we have \(Y_{i,t_{ij}} = Y_{i,t_j}\). If \(t_{ij} \notin \mathcal{F}\), the respective synchronous log price \(Y_{i,t_j}\) is considered as a missing value and will be filtered by the HFF synchronizing technique presented in Section 3.1.

### 2.3 Realized covariance estimators

In this subsection, we will briefly introduce the existing estimators of the integrated volatility matrix and the synchronizing techniques employed. They are the pre-averaging estimator with the synchronous method of Hayashi and Yoshida (denoted by \(\hat{\Sigma}_p + HY\)), the kernel estimator with the refresh time method (denoted by \(\hat{\Sigma}_k + RT\)), the two-scaled estimator with the previous tick method (denoted by \(\hat{\Sigma}_t + PT\)).
Figure 1 illustrates how the previous tick (PT) and refresh time (RT) technique synchronize data. The PT takes the last available observation if the log price is missing at any particular time point. The RT only considers time points when all assets have observations since last time. While the PT may have to repeat the previous log price many times and hence distort the dependence structure in the assets, the RT can discard much information that are eventually useful.

In the following successive subsections, we denote the same notation \((\tau_0, \tau_1, \ldots, \tau_{n_0})\) to represent the time points by using any synchronized techniques (PT or RT) without any confusing. The corresponding log prices are denoted by \((Y_{\tau_0}, Y_{\tau_1}, \ldots, Y_{\tau_{n_0}})\) with \(Y_{\tau_j} = (Y_{1,\tau_j}, \ldots, Y_{p,\tau_j})\) for \(j = 0, 1, \ldots, n_0\).

Figure 1: Previous tick (left) and refresh time (right) synchronizing techniques. A vertical dash (\(|\)) represents an observation of the asynchronous and noisy log price processes \(Y_1\) to \(Y_3\). A cross (\(\times\)) represents sampling point of the resulting synchronous processes.
2.3.1 Pre-averaging estimator with the synchronous method of Hayashi and Yoshida $\hat{\Sigma}_p + HY$

Given the function $g(x) = \min(x, 1-x)$ and $k_n = [\theta \sqrt{n_0}]$, the pre-averaging estimator is:

$$\hat{\Sigma}_p = \frac{n}{n - k_n + 2 k_n} \sum_{j=0}^{n-k_n+1} \bar{Y}^n_{\tau_j} (\bar{Y}^n_{\tau_j})^* - \frac{(12)^{k_n}}{2 n_0 \theta^2} \sum_{j=1}^{n_0} (Y_{\tau_j} - Y_{\tau_{j-1}}) (Y_{\tau_j} - Y_{\tau_{j-1}})^*,$$ \hspace{1cm} (2.5)

where

$$\bar{Y}^n_{\tau_j} = \frac{1}{k_n} \left( \sum_{\ell = k_n/2}^{k_n-1} Y_{\tau_j + \ell} - \sum_{\ell = 0}^{k_n/2} Y_{\tau_j + \ell} \right)$$

and the last term of (2.5) is a bias-correction term. Christensen et al. (2010) show that $\hat{\Sigma}_p$ is an unbiased estimator of $\Sigma$ with convergence rate $O_p(n^{-1/4})$. However, this estimator $\hat{\Sigma}_p$ cannot ensure semi-positive definition. By taking $k_n = [\theta n_0^{0.6}]$, the bias-correction term can be ignored and the estimator becomes SPD, but the convergence rate will reduce to $O_p(n^{-1/5})$.

Furthermore, for the asynchronous case, Christensen et al. (2010) also proposed a covariance estimator by using the procedure of Hayashi and Yoshida (2005). Set $n = \sum_{k=1}^p n_k$ and $k_n = [\theta \sqrt{n}]$, the $(k, m)$ element of the covariance estimator is

$$\frac{1}{(k_n/4)^2} \sum_{i=0}^{n_k-k_n+1} \sum_{j=0}^{n_m-k_m+1} \bar{Y}^n_{k,s_k,i} \bar{Y}^n_{m,s_m,j} I\{ (s_{k,i}, s_{k,i}+k_n) \cap (s_{m,j}, s_{m,j}+k_n) \neq \phi \},$$ \hspace{1cm} (2.6)

where

$$\bar{Y}^n_{k,s_k,i} = \frac{1}{k_n} \left( \sum_{\ell = k_n/2}^{k_n-1} Y_{k,s_k,i + \ell} - \sum_{\ell = 0}^{k_n/2} Y_{k,s_k,i + \ell} \right).$$
2.3.2 Kernel estimator with the refresh time method $\hat{\Sigma}_k + RT$

For a given kernel function $k(x)$, the multivariate realized kernels is

$$\hat{\Sigma}_K = \sum_{h=-n_0}^{n_0} k\left(\frac{h}{H}\right) \Gamma_h, \quad (2.7)$$

where

$$\Gamma_h = \sum_{j=h+1}^{n} (Y_{\tau_j} - Y_{\tau_{j-1}})(Y_{\tau_{j-h}} - Y_{\tau_{j-h-1}})^*, \quad \text{for } h \geq 0,$$

and $\Gamma_h = \Gamma_{-h}$ for $h < 0$. Barndorff-Nielsen et al. (2011) shown that if $H = c_0 n^{3/5}$ the multivariate realized kernels $\hat{\Sigma}_K$ is an unbiased estimator of $\Sigma$ with convergence rate $O_p\left(n^{-1/5}\right)$. The estimator $\hat{\Sigma}_K$ is an SPD matrix.

2.3.3 Two-scaled estimator with the previous tick method $\hat{\Sigma}_t + PT$

For a fixed constant $m$ and $K = \lfloor n_0/m \rfloor$, the two-scaled estimator is defined as

$$\hat{\Sigma}_T = \frac{1}{K} \sum_{\kappa=1}^{K} \sum_{j=1}^{m-\kappa} (Y_{\tau_{jK+(\kappa-1)}} - Y_{\tau_{jK+(\kappa-1)}})(Y_{\tau_{jK+(\kappa-1)}} - Y_{\tau_{jK+(\kappa-1)}})^* - 2m\hat{\eta}, \quad (2.8)$$

where $\hat{\eta} = \text{diag}(\hat{\eta}_1, \ldots, \hat{\eta}_p)$ with $\hat{\eta}_i = \frac{1}{2n_0} \sum_{j=1}^{n_0} (Y_{i,\tau_j} - Y_{i,\tau_{j-1}})^2$. Wang and Zou (2010) show that for $K = O(n^{2/3})$, the two-scaled estimator $\hat{\Sigma}_T$ is an unbiased estimator of $\Sigma$ with convergence rate $O_p\left(n^{-1/6}\right)$. However the estimator $\hat{\Sigma}_T$ cannot ensure the semi-positive definition.
3 The efficient SPD estimator

Now we present the algorithm of the HFF synchronizing technique in Section 3.1. Compared to other techniques, the HFF technique can obtain HF data with large sample size \( n \) and hence enhance e.g. the pre-averaging estimator’s efficiency at \( O_p(n^{-1/4}) \). We also provide a general correction approach to obtain an integrated SPD covariance matrix estimator, which has the same limiting distribution as the efficient estimator and reaches the optimal convergence rate. The proposed SPD estimator and its asymptotic properties will be presented in Section 3.2.

3.1 The HFF synchronizing technique

We present the High Frequency Filtering (HFF) synchronizing technique. Given the asynchronous and noisy HF data, \( Y_{i,t_{ij}} \) with \( t_{ij} \in \mathcal{F} \), it is to recursively filter the missing values at \( t_{ij} \notin \mathcal{F} \) and obtain synchronous but still noisy log prices \( \hat{Y}_{t_j} = (\hat{Y}_{t_1}, \ldots, \hat{Y}_{t_p})^*, j = 0, \ldots, n \). Without loss of generality, we assume that the initial value of each component data \( Y_{i,t_0} \) exists for all \( i = 1, \ldots, p \), i.e. \( \hat{Y}_{t_0} = Y_{t_0} \).

Suppose a preliminary covariance estimator \( S_0 \) is available, which reflects the dependence information of the noisy data. It is eventually an estimator of the integrated covariance matrix \( \Sigma \) plus the microstructure noise variance. We have the spectral decomposition of \( S_0 \) as:

\[
S_0 = \Gamma \hat{A} \Gamma^* = \sum_{i=1}^{p} \hat{a}_i \gamma_i \gamma_i^*
\]  

(3.9)

where \( \hat{A} \) is a diagonal matrix with eigenvalues \( \hat{a}_i \) of \( S_0 \) on the diagonal, \( \Gamma \) is the associated orthonormal eigenvectors matrix.
Assume there exists a linear filter $Z_{tj} = (Z_{1tj}, \ldots, Z_{ptj})$ that is a projection of the unobserved synchronous log returns $R_{tj} = Y_{tj} - Y_{tj-1}$:

$$R_{tj} = \Gamma Z_{tj}, \quad t_j = j/n, \quad j = 1, \ldots, n.$$  \hspace{1cm} (3.10)

where $\Gamma$ is the eigenvector matrix in (3.9). The linear filter is synchronous and employs the dependence information of the original data.

Starting from time $t_1$ to $t_n$, the HFF technique iteratively synchronize data at high sampling frequency with a sequential procedure. At any time $t_j$, if $t_{ij} \notin \mathcal{F}$ for all $i = 1, \ldots, p$, we have the previous log price $\hat{Y}_{tj} = \hat{Y}_{tj-1}$; otherwise $\hat{Y}_{tj}$ is the filtered log price as the following procedure. Denote $\hat{R}_{i,tj} = Y_{i,tj} - \hat{Y}_{i,tj-1}$ to be the observed log returns for the case that $t_{ij} \in \mathcal{F}$. We are searching a linear filter $Z_{tj}$ that minimizes the distance between the observed and filtered log returns and considers the lag-1 autocorrelation of the noisy data. The objective function is defined:

$$\min_{Z_{tj}} \left\{ \sum_{i=1}^{P} \left[ \left( \hat{R}_{i,tj} - \gamma_i^* Z_{tj} \right)^2 I\{t_{ij} \in \mathcal{F}\} \right] + \delta (Z_{tj} + 0.5Z_{tj-1})^* \hat{A}^{-1} (Z_{tj} + 0.5Z_{tj-1}) \right\},$$  \hspace{1cm} (3.11)

where the first part minimizes the projection accuracy (3.10). The second part reflects the existence of lag-1 autocorrelation that also ensures the synchronizing technique continues, even if the concurrent returns of some assets are not observed. We standardize the filtered series by its own variance, the eigenvalues of $\hat{A}$. The tuning parameter $\delta$ controls the level of smoothness of the filtered series. While large values of $\delta$ lead to over-smooth, small values may create unnecessarily rough process. Cross-validation is used to select the optimal value of $\delta$. Finally, the filtered log price is obtained by $\hat{Y}_{tj} = \Gamma \hat{Z}_{tj} + \hat{Y}_{tj-1}$ where $\hat{Z}_{tj}$ is the solution of (3.11).

The formal algorithm of the HFF technique is presented as follows:
Set \( j = 1 \). Let \( Z_{t_0} = 0 \) and we have \( S_0 = \Gamma \hat{A} \Gamma^* \).

1. If \( t_{ij} \notin \mathcal{F} \) for all \( i = 1, \ldots, p \), set \( \hat{Y}_{t_j} = \hat{Y}_{t_{j-1}} \) and jump to step 4.

2. If \( t_{ij} \in \mathcal{F} \) with at least one \( i = 1, \ldots, p \), compute log return \( \hat{R}_{i,t_{ij}} = Y_{i,t_{ij}} - \hat{Y}_{i,t_{j-1}} \) for every \( i \) satisfying \( t_{ij} \in \mathcal{F} \).

3. Obtain the linear filter \( \hat{Z}_{t_j} \) that minimizes the objective function (3.11). We have \( \hat{Y}_{t_j} = \Gamma \hat{Z}_{t_j} + \hat{Y}_{t_{j-1}} \).

4. Stop until \( j = n \); otherwise renew \( j = j + 1 \) and return to step 1.

In our study, the pre-averaging estimator described on Section 2.3 plus the estimator of variation of noise in Wang and Zou (2010) is used as the preliminary estimator \( S_0 \) (3.9). The pre-averaging estimator is however computed with low frequency observations, which is consistent but not efficient.

### 3.2 The SPD estimator

For this moment, suppose we have an integrated covariance estimator, denoted as \( S_1 \), which is computed with synchronous log prices \( Y_{t_j}, j = 0, \ldots, n \). Although \( \Sigma \) is a semi-positive definite (SPD) matrix with \( \Sigma \geq 0 \), or a positive definite matrix with \( \Sigma > 0 \), the estimator \( S_1 \) may not satisfy the condition of \( S_1 \geq 0 \) or \( S_1 > 0 \). We propose a general approach to construct a non-negative definite estimator \( S \) that has the same convergence rate and limiting distribution as the preliminary estimator \( S_1 \).

Denote the spectral decompositions of \( S_1 \) and \( \Sigma \) by

\[
S_1 = U \hat{A} U^* = \sum_{i=1}^P \hat{\lambda}_i u_i u_i^*, \quad \Sigma = V AV^* = \sum_{i=1}^P \lambda_i v_i v_i^* \quad (3.12)
\]

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where \(\hat{\lambda}'s\) and \(\lambda's\) are respectively eigenvalues of \(S_1\) and \(\Sigma\), and \(u_i\) and \(v_i\) are orthonormal eigenvectors associated with \(\hat{\lambda}_i\) and \(\lambda_i\) for all \(i\). Our SPD estimator, denoted by \(S\), of \(\Sigma\), is proposed as:

\[
S = U|\hat{\Lambda}|U^*,
\]

where \(|\hat{\Lambda}| = \text{diag}(|\hat{\lambda}_1|, \ldots, |\hat{\lambda}_p|)\), and \(U\) and \(\hat{\Lambda}'s\) are defined in (3.12). Theorem 3.1 below shows that \(S\) is a consistent estimator and we also derive the asymptotic distribution of \(S\) in Theorem 3.2.

Before proving Theorem 3.1, we cite the following lemma from Bai, Miao and Rao (1991) for readers’ convenience.

**Lemma 3.1** (Bai et al., 1991)

Let \(A = (a_{ik})\) and \(B = (b_{ik})\) be two Hermitian \(p \times p\) matrices with spectral decompositions

\[
A = \sum_{i=1}^{p} \delta_i u_i u_i^*, \quad \delta_1 \geq \delta_2 \geq \cdots \geq \delta_p,
\]

and

\[
B = \sum_{i=1}^{p} \lambda_i v_i v_i^*, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p,
\]

where \(\delta's\) and \(\lambda's\) are eigenvalues of \(A\) and \(B\), respectively, \(u_i's\) and \(v_i's\) are orthonormal eigenvectors associated with \(\delta's\) and \(\lambda's\), respectively. Further, we assume that

\[
\lambda_{n_{b-1}+1} = \cdots = \lambda_{n_b} = \hat{\lambda}_b, \quad n_0 = 0 < n_1 < \ldots < n_r = p, \quad b = 1, \ldots, r,
\]

\[
\hat{\lambda}_1 > \hat{\lambda}_2 > \cdots > \hat{\lambda}_r,
\]

and that

\[
|a_{ik} - b_{ik}| < \alpha, \quad i, k = 1, \ldots, p.
\]
Then there is a constant $M$ independent of $\alpha$, such that

(a) $|\delta_i - \lambda_i| < M\alpha$, \quad $i = 1, \ldots, p$.

(b) $\sum_{i=n_b-1+1}^{n_b} u_i u_i^* = \sum_{i=n_b-1+1}^{n_b} v_i v_i^* + C^{(b)}$,

where $C^{(b)} = (C^{(b)}_{ik})$, $|C^{(b)}_{ik}| < M\alpha$, $l, k = 1, \ldots, p$, $b = 1, \ldots, r$.

**Theorem 3.1** Suppose that $\Sigma \geq 0$ and the maximum eigenvalue of $\Sigma$, denoted by $\lambda_{\max}$, is bounded. Let $S_1$ be a symmetric matrix satisfying

$$S_1 - \Sigma \xrightarrow{P} 0. \quad (3.14)$$

Define $S$ by (3.13). Then $S$ is a consistent estimator of $\Sigma$, that is,

$$S - \Sigma \xrightarrow{P} 0.$$

**Proof.** By (3.14),

$$(S_1^2 - \Sigma^2) = (S_1 - \Sigma)^2 + (S_1 - \Sigma)\Sigma + \Sigma(S_1 - \Sigma), \quad (3.15)$$

we obtain that $S_1^2 - \Sigma^2$ converges to zero in probability. Assume that there are $r$ different values of $(\lambda_1, \ldots, \lambda_p)$, say

$$\lambda_{n_b-1+1} = \cdots = \lambda_{n_b} = \tilde{\lambda}_b, \quad n_0 = 0 < n_1 < \ldots < n_s = p, \quad b = 1, \ldots, r.$$
By Lemma 3.1 and (3.15), we have for any $\epsilon > 0$,

$$|\hat{\lambda}^2 - \lambda^2| < M\epsilon, \text{ } i = 1, \ldots, p, \text{ in probability}$$

$$\sum_{i=n_{b-1}+1}^{n_b} u_i u_i^* - \sum_{i=n_{b-1}+1}^{n_b} v_i v_i^* = C^{(b)}$$

where $C^{(b)} = (C^{(b)}_{lk})$ with $|C^{(b)}_{lk}| \leq M\epsilon$ in probability for all $l, k = 1, \ldots, p$ and $b = 1, \ldots, r$.

It follows from (3.16), (3.17) and Slutsky’s Theorem that

$$S - \Sigma = \sum_{i=1}^{p} \left( |\hat{\lambda}| u_i u_i^* - \lambda_i v_i v_i^* \right) = \sum_{b=1}^{r} |\hat{\lambda}_b| \sum_{i=n_{b-1}+1}^{n_b} u_i u_i^* - \sum_{b=1}^{r} \lambda_b \sum_{i=n_{b-1}+1}^{n_b} v_i v_i^*$$

$$= \sum_{b=1}^{r} \left( |\hat{\lambda}_b| - \lambda_b \right) \sum_{i=n_{b-1}+1}^{n_b} (u_i u_i^*) + \sum_{b=1}^{r} \lambda_b \sum_{i=n_{b-1}+1}^{n_b} (u_i u_i^* - v_i v_i^*)$$

$$\leq M\epsilon \text{ in probability},$$

which implies $S - \Sigma \overset{P}{\rightarrow} 0$. \hfill \Box

When reinforcing the condition on $\Sigma$, we conclude that $S$ and $S_1$ have the same limiting distribution as shown in the following theorem.

**Theorem 3.2** Suppose that $\Sigma > 0$ and $|\lambda_{\text{max}}|$ is bounded. Let $S_1$ be a symmetric matrix satisfying

$$\alpha_n (S_1 - \Sigma) \overset{d}{\rightarrow} Z,$$

where $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. Define $S$ by (3.13). Then

$$\alpha_n (S - \Sigma) \overset{d}{\rightarrow} Z.$$
Proof. By (3.19) and Slutsky’s Theorem, we have

\[ \alpha_n(S_1^2 - \Sigma^2) = \alpha_n(S_1 - \Sigma)^*(S_1 - \Sigma) + \alpha_n(S_1 - \Sigma)^*\Sigma + \alpha_n\Sigma(S_1 - \Sigma) \]

\[ = \alpha_n^{-1} [\alpha_n(S_1 - \Sigma)]^* [\alpha_n(S_1 - \Sigma)] \]

\[ + [\alpha_n(S_1 - \Sigma)]^* \Sigma + \Sigma [\alpha_n(S_1 - \Sigma)] \]

\[ \xrightarrow{d} Z\Sigma + \Sigma Z. \quad (3.20) \]

This implies \( S_1^2 - \Sigma^2 = O_p(\alpha_n^{-1}) \). By Lemma 3.1 we then have

\[ |\hat{\lambda}_i^2 - \lambda_i^2| < M\alpha_n^{-1}, \quad i = 1, \ldots, p, \text{ in probability} \quad (3.21) \]

\[ \sum_{i=n_{b-1}+1}^{n_b} u_i u_i^* - \sum_{i=n_{b-1}+1}^{n_b} v_i v_i^* = C^{(b)} \quad (3.22) \]

where \( C^{(b)} = (C^{(b)}_{lk}) \) with \( |C^{(b)}_{lk}| \leq M\alpha_n^{-1} \text{ in probability} \) for all \( l, k = 1, \ldots, p \) and \( b = 1, \ldots, r \). Then, by (3.21), (3.22), Slutsky’s Theorem and an argument similar to (3.18), we have

\[ S - \Sigma \leq M'\alpha_n^{-1} \text{ in probability}, \quad (3.23) \]

where \( M' \) is some positive constant independent of \( n \).

The condition (3.23) implies that \( \alpha_n(S - \Sigma) \) is tight. Consider a subsequence \( n_k \) on which \( \alpha_{n_k}(S - \Sigma) \) converges in distribution to a random variable, say \( Y \) (here and below, to save notation we still use \( S - \Sigma \) rather than their expressions on the subsequence). Therefore, by Slutsky’s Theorem, it holds that

\[ \alpha_{n_k}(S^2 - \Sigma^2) = \alpha_{n_k}(S - \Sigma)S + \alpha_{n_k}\Sigma(S - \Sigma) \xrightarrow{d} Y\Sigma + \Sigma Y. \quad (3.24) \]
Evidently, (3.24) is equivalent to

$$\text{vec}(\alpha_n (S^2 - \Sigma^2)) \overset{d}{\to} (I \otimes \Sigma + \Sigma \otimes I) \text{vec}(Y),$$

where $\otimes$ is the Kronecker product and $\text{vec}(Y)$ is the vectorization of a matrix.

Moreover, (3.20) can be also rewritten as

$$\text{vec}(\alpha_n (S_1^2 - \Sigma^2)) = \text{vec}(\alpha_n (S^2 - \Sigma^2)) \overset{d}{\to} (I \otimes \Sigma + \Sigma \otimes I) \text{vec}(Z).$$

Note that the eigenvalues of $(I \otimes \Sigma + \Sigma \otimes I)$ are $\{\lambda_i + \lambda_j, \ i, j = 1, \ldots, p\}$. The hypothesis $\Sigma > 0$ implies that $(I \otimes \Sigma + \Sigma \otimes I)$ is invertible. Therefore, (3.25) and (3.26) ensure that $\text{vec}(Z) \overset{d}{=} \text{vec}(Y)$ and hence $Z \overset{d}{=} Y$. This means that $Y$ is unique. This, together with the tightness of $\alpha_n (S - \Sigma)$ (which implies that $\alpha_n (S - \Sigma)$ is relatively compact), implies that $\alpha_n (S - \Sigma) \overset{d}{\to} Z$.

**Remark 3.1** For the preliminary estimator $S_1$, we can refer to Christensen et al. (2010) which has the optimal convergence rate $O_p(n^{-1/4})$ but cannot make sure that the estimated matrix is semi-positive definite. Although Christensen et al. (2010) propose an SPD estimator, the convergence rate is reduced to $O_p(n^{-1/5})$. Other estimators, see for example, Wang and Zou (2010), Barndorff-Nielsen et al. (2011), have the convergence rates slower than the optimal one. In next subsection, we will use simulations to show that the proposed estimator $S$ has the same limiting distribution as that of Christensen et al. (2010) possessing the optimal convergence rate.
4 Simulation study

We consider a series of simulation studies to investigate the finite sample performance of the proposed methods. We first generate the synchronous HF data to confirm the limiting distributional property of our proposed estimator, as described in Section 4.1. Next Section 4.2 compares the features of our proposed estimator to several alternatives.

4.1 Simulation: synchronous data

As shown in Section 3.2, our proposed estimator has the same limiting distribution as the preliminary estimator $S_1$ that is efficient but may not be SPD. In the simulation, we follow the model setting in Wang and Zou (2010) and generate synchronous data. The log price $X_t$ of $p$ assets is generated from

$$dX_t = \sigma_t^* dB_t, \quad t \in [0, 1],$$

where $B_t = (B_{1t}, \ldots, B_{pt})^*$ is a standard $p$-dimensional Brownian motion and $\sigma_t$ is a Cholesky decomposition of $\Sigma_t = (\Sigma_{ij,t})_{1 \leq i, j \leq p}$ which is defined below. Let the diagonal elements of $\Sigma_t$ follow a CIR process, that is

$$d\Sigma_{ii,t} = \theta_i (\mu_i - \Sigma_{ii,t})dt + \omega_i \sqrt{\Sigma_{ii,t}}dW_{it},$$

where $\mu_i$ denotes the long term mean of the volatility, $i = 1, \ldots, p$, and $W_{it}$ are standard one dimensional Brownian motion independent of $B_t$. Define the off-diagonal
elements by

\[ \Sigma_{ij,t} = [\kappa(t)]^{i-j} \sqrt{\Sigma_{ii,t} \Sigma_{jj,t}}, \quad 1 \leq i \neq j \leq p, \]

where \( \kappa(t) \) is given by

\[ \kappa(t) = \frac{e^{2u(t)} - 1}{e^{2u(t)} + 1}, \quad du(t) = 0.3[0.64 - u(t)]dt + 0.118u(t)dW_{\kappa,t}, \]

\[ W_{\kappa,t} = \sqrt{0.96W_{\kappa,t}^0 - 0.2 \sum_{i=1}^{p} B_{it}/\sqrt{p}}, \]

and \( W_{\kappa,t}^0 \) is standard one dimensional Brownian motion independent of \( B_t \) and \( W_{it}. \)

We generate the synchronous but noisy log price as follows:

\[ Y_{t_j} = X_{t_j} + \epsilon_{t_j}, \]

where \( t_j = j/n \) with \( j = 0, \ldots, n, \) and \( \epsilon \) is IID random vector with mean zero and finite second moments.

We experiment on the following parameter setting: \( p = 5, \ \eta_i = 4 \times 10^{-7}, \ i = 1, \ldots, 5, \ (\mu_1, \ldots, \mu_5) = (4 \times 10^{-5}, 1.6 \times 10^{-5}, 1.6 \times 10^{-4}, 4 \times 10^{-5}, 1.6 \times 10^{-5}), \ \theta_1 = \cdots = \theta_5 = 10, \ \omega_i = \sqrt{\mu_i \theta_i}/2, \) sample size \( n = 23400 \) and the replications are 1 000 times. For the choice of the preliminary estimator \( S_1, \) we use the pre-averaging estimator as shown in Christensen et al. (2010). Note that this parameter setting may not match the realistic situation, but easily leads to negative definite covariance matrix estimators. Among the 1 000 times replications, there are 222 times of negative definite covariance estimators. Given the preliminary estimators, we obtain the SPD estimators \( S \) in (3.13). The Chi-square goodness-of-fit statistic is used to test the null hypothesis that all the elements of \( n^{1/4}(S_1 - \Sigma) \) and \( n^{1/4}(S - \Sigma) \) have the same
distribution. The result of the \( p \)-values is

\[
\begin{pmatrix}
1 & 0.99 & 1 & 1 & 0.99 \\
0.99 & 0.72 & 1 & 0.99 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0.99 & 1 & 1 & 0.99 \\
0.99 & 0.99 & 1 & 0.99 & 0.99
\end{pmatrix}
\]

which represents a strong evidence to accept that \( S_1 \) and \( S \) have the same limiting distribution.

### 4.2 Simulation: asynchronous data

In this section, we use the same model as shown in Section 4.1 but construct asynchronous and noisy observations by controlling five Poisson processes with intensities \( \psi = (\psi_1, \ldots, \psi_5)^* \). We compare the eigenvalues of our proposed efficient SPD estimator with high frequency filtration technique (denoted by \( \hat{\Sigma}_s + HFF \)) to the alternatives stated in Section 2.3. The tuning parameter \( \delta \) is set to be \( 10^{-4} \).

We first consider three practically oriented experiments based on the assets in finance, electronic and food sectors at NYSE. In each sector, five assets are selected. The parameters and intensities are respectively estimated with the real data, see Table 1. The generated processes on average have \( 23400/\psi_1 \) to \( 23400/\psi_5 \) observations. Each experiment is named after the respective sector. In addition, we consider one Noisy experiment with slight signals against vast noise, an extremely asynchronous (Ex-Asy) experiment with very dissimilar sampling frequency of the five assets, and an experiment with high sampling frequency of all the five assets denoted Ex-HF.
Table 1: The parameter setting of the long term mean of the volatility ($\mu_i$), the variance of the microstructure noise ($\eta_i$) and the intensities ($\psi_i$). The number in bold-face indicates the best accuracy in each experiment.

<table>
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<tr>
<th>$i$</th>
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<th></th>
<th>Noisy</th>
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<td></td>
<td>1 2 3 4 5</td>
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<td>$\mu_i$ ($\times 10^{-4}$)</td>
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For the comparison, we define the relative errors of five eigenvalues as

$$RE_i = \sqrt{\frac{1}{m} \left[ \sum_{s=1}^{m} (\hat{\lambda}_i^{(s)} - \lambda_i)^2 \right] / \lambda_i}, \quad i = 1, \ldots, 5,$$

where $\lambda_i$ is the true eigenvalues at dimension $i$ and $\hat{\lambda}_i^{(s)}$ is the estimated eigenvalues at dimension $i$ in the $s$–th replication, $s = 1, \ldots, m = 1000$. Table 2 reports the average relative errors of the five eigenvalues from the four covariance estimators. In each experiment, the best accurate result is marked in bold. It shows that the proposed efficient SPD estimator, denoted by $\hat{\Sigma}_n + HFF$, has almost the smallest relative errors compared to the alternatives.

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Moreover, the computational time of $\hat{\Sigma}_{p} + HY$, $\hat{\Sigma}_{k} + RT$, $\hat{\Sigma}_{t} + PT$ and $\hat{\Sigma}_{s} + HFF$ for one replication are 178, 14, 20, 62 (sec.), respectively.

Besides, we also investigate the sensitivity of the tuning parameter $\delta$. In several experiments on the different values of the tuning parameter $\delta$, we see that the covariance estimators are almost the same as $\delta \in [10^{-5}, 10^{-4}]$, which is the quantities of the largest eigenvalues. Thus, the $\delta$ is chosen such that the second part of (3.11) has the same order as the first part of (3.11).

5 Conclusion

We develop a new realized covariance estimator that ensures semi-positive definition and optimal efficiency simultaneously. By learning from the dependence information of the data, we are able to iteratively synchronize the asynchronous HF data. Together with a correction approach, the developed estimator is semi-positive definite and efficient at the optimal convergence rate $O_p \left( n^{-1/4} \right)$. It is consistent and has the same limiting distribution as the efficient estimator. Real data oriented simulation experiments demonstrates the finite sample performance of the estimator. Compared to several alternatives, the efficient SPD estimator also provides the best accuracy in various experiments.

References


Table 2: The relative errors of five eigenvalues.

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