Slope reliability analysis: some insights and guidance for practitioners

B. K. Low
School of Civil and Environmental Engineering
Nanyang Technological University, Singapore
E-mail: bklow@alum.mit.edu

ABSTRACT: Several topics of concern have emerged in recent literature on probabilistic slope stability analysis, including the relative merits of probabilistic finite element methods, probabilistic limit equilibrium methods, Monte Carlo simulations, and first-order reliability method. Practitioners contemplating extending their long tradition of deterministic analysis into probabilistic analysis may be perplexed and even discouraged by the different views of the researchers of different probabilistic approaches. It is the purpose of this paper to contribute some personal insights and guidance on slope reliability analysis, and to provide a more balanced perspective of the different probabilistic slope stability approaches. Specifically, the following topics will be addressed, mainly in the context of soil and rock slopes: (i) Avoiding potential pitfalls in using Monte Carlo simulations, (ii) FORM reliability-based design, (iii) Probabilistic slope stability analysis accounting for spatial variation of soil properties, and (iv) System reliability analysis based on FORM and comparison with MC simulations.

1. INTRODUCTION

Different deterministic and probabilistic approaches (e.g. FEM, LEM, Monte Carlo simulations, FORM, etc.) are valuable and can contribute in their complementary roles. Like all scientific/engineering methods/models, each approach has strengths and limitations, and no one method is perfect for all problems. The insights and guidance in this paper are based on personal experience, and given with the sole objective of enhancing understanding and avoiding potential pitfalls.

2. MONTE CARLO: VALUABLE, IF USED PROPERLY

Monte Carlo simulation can be used to estimate the probabilities of failure of a system, or to provide comparative analysis to reliability methods. Care and understanding are however needed to avoid potential pitfalls, two of which are explained below.

2.1 Distortions caused by negative values of random numbers

Figure 1 shows a spring of rupture strength \( Q_u \), suspending a vertical load of magnitude \( Q \), both in units of force. The mean values and standard deviations of \( Q_u \) and \( Q \) are shown for two cases, A and B, respectively, in which the uncertainty of the applied load \( Q \) is bigger in Case B than in Case A (hence larger standard deviations of \( Q \) in case B). The two random variables are assumed to be independent for illustration. One can use the first-order reliability method (FORM), which includes the Hasofer-Lind index as a special case, to obtain the reliability index \( \beta \). The classical FORM requires mathematical techniques for rotated and transformed space. Figure 1 uses the Low & Tang (2007) procedure of spreadsheet-automated constrained optimization approach that obtains the same design point and reliability index as the classical FORM, but more intuitively and directly and on the ubiquitous spreadsheet platform.

The performance function (or limit state function) for the problem in Fig. 1 can be expressed in two ways which are physically equivalent when \( g(x) = 0 \):\n
\[
g_1(x) = Q_u - Q \quad \text{or} \quad g_2(x) = \frac{Q_u}{Q} - 1 \quad (1a); (1b)
\]

Using FORM, one obtains identical value of reliability index \( \beta \) (2.236 for Case A and 1.562 for Case B) regardless of whether Eq. (1a) or Eq. (1b) is used. The corresponding probabilities of failure are 1.27% for Case A and 5.92% for Case B, respectively. Using Monte Carlo simulations with Latin Hypercube sampling (each 500,000 trials), the probabilities of failure (\( P_f \)) based on performance functions (1a) and (1b) are identical (in the range 1.24%~1.29%) and consistent with that based on FORM’s \( \Phi(-\beta) \) only for Case A. For case B, Monte Carlo simulations produce results (5.90%~5.94%) consistent with FORM’s result of 5.92% only if the performance function of Eq. (1a) is used. When the performance function of Eq. (1b) is used, Monte Carlo simulations yield a misleading \( P_f \) of 8.18%~8.22% versus the correct result of 5.92%.

The wrong results of Monte Carlo for \( g(x) \) of case B is due to the coefficient of variation of \( Q \) being 25/50=0.5, which means that about 2.275% of the 500,000 random sets in each simulation fall in the negative range of the lower tail (at greater than two standard deviations from the mean value of \( Q \)). These negative random numbers of \( Q \) do not distort \( g(x) \) of Eq. (1a), which remains positive. However, negative random numbers of \( Q \) render Eq. (1b) negative, since the \( Q_u \) in the numerator is virtually always positive during MC simulations, being five standard deviations away from the negative range. The result is: the correct 5.92% failure rate + a phantom 2.275% = erroneous 8.20%±. It helps to understand that a small positive load close to zero will yield large positive value of \( g(x) \) of Eq. (1b), signifying safety; negative loads (which can happen in MC simulations) should logically be even safer.

![Figure 1](image-url)
The above inaccuracies in Monte Carlo simulations can be avoided easily by (i) expressing the performance functions in the form of Eq. (1a), or (ii) using probability distributions which exclude negative domain, e.g., lognormal, truncated normal, or the bounded 4-parameter general beta distribution. Of course, when the coefficient of variation of the denominator in Eq. (1b) is small such that negative random numbers are virtually impossible during MC simulations (as in Case A), performance functions in the form of Eq. (1b) will yield results as reliable as Eq. (1a). Case B in Figure 1 should not be construed as discouraging the use of normal distribution, merely that one needs to check the effect on the performance function if Monte Carlo simulation generates negative numbers. Unlike Monte Carlo simulations, the design point values of FORM (under the $x_i^*$ columns in cases A and B) can be observed easily to check their legitimacy. There is nothing illegitimate about the design point values of 60 (for case A) and 80.488 (for case B).

2.2 Distortions caused by physically incompatible random numbers in Monte Carlo simulations

The admissible physical relationship between some random variables must not be violated if correct results are to be obtained. Consider the case of a two-dimensional rock slope shown in Fig. 2. A correct treatment of the two random variables $z$ and $z_s$ is described first in the next paragraph, followed by incorrect treatment in the paragraph after.

The tension crack is of depth $z$, and filled with water to the depth $z_s$. From physical considerations, the minimum value for $z_s/z$ is 0, when the tension crack is dry. The maximum value is 1, when the tension crack is completely filled with water. In the FORM reliability analysis and Monte Carlo simulations of Low (2007), both $z$ and $z_s$ will change from their mean values, but both must be restricted to the domain $0 \leq z_s/z \leq 1$. Normal distribution was assumed for $z$, and truncated exponential (trimmed_exp) probability distribution was assumed for $z_s/z$. For the case without reinforcement ($T = 0$), the computed reliability index $\beta$ was 1.556, corresponding to a failure probability of about 6.0%, obtained from $\Phi_i = \Phi(-\beta)$. This compares reasonably well with the failure rates of 6.8%, 6.2%, 6.5%, 6.5%, 6.6%, 6.4% from six Monte Carlo simulations (each with a sample size of 5000 based on Latin Hypercube sampling using the commercial software @RISK, http://www.palisade.com). The difference between the reliability-index-inferred probability of failure of 6% and the average of 6.5% from Monte Carlo simulations is due to the approximate nature of the equivalent normal transformation in FORM when nonnormals are involved, and possible nonlinearity in the limit state function $g(x) = [F_\phi, F_z, z_s/z, \alpha] - 1$, which render the equation $\Phi_i = \Phi(-\beta)$ approximate. (The equation is exact when the random variables are normally distributed and the limit state surface is planar, as in the cases of Fig. 1.)

Instead of modelling $z$ and $z_s/z$ as two random variables (with the mean value of the latter equal to 0.5), one could conceivably treat $z$ and $z_s$ as two random variables, with the mean value of $z$ equal to 14 m, and that of $z_s$ equal to 7 m. However, doing so leads to paradoxical values of probability of failure from Monte Carlo simulations which are several times greater than the average of 6.5% (also from Monte Carlo simulations) mentioned in the preceding paragraph when $z$ and $z_s/z$ were modelled. Examination of the range of numbers of $z$ and $z_s/z$ generated in MC simulations (when $z$ and $z_s$ were modelled separately) revealed that there were many failure cases (performance function $g(x) < 0$) involving generated $z_s$ values greater than $z$ values, which are physically inadmissible, but happened in the random numbers generated during MC simulations.

One can think of two other random variables whose physical dependency on each other should not be violated in Monte Carlo simulations if correct results are to be obtained: the friction angle $\phi'$ of retained backfill and the interface friction angle $\delta$ between the retained fill and the retaining wall. Physical considerations require $\delta \leq \phi'$. If $\delta$ values greater than $\phi'$ values are generated in MC simulations together with positive performance function values, the probability of failure from MC simulation could be underestimated.

3. ADVANTAGES AND SUBTLETIES IN RELIABILITY-BASED DESIGN

The merits of reliability-based design are illustrated for the simple case of the spring-suspended load of Fig. 1. The need to distinguish negative from positive reliability index in reliability-based design is then discussed in the context of the rock slope of Fig. 2.

3.1 Context-specific load and resistance factors in reliability-based design

The computed values of the reliability index $\beta$ for the loaded spring in Fig. 1 are 2.236 and 1.562 when the values of the coefficient of variation, $\sigma/\mu$, of the applied load $Q$ are 0.2 and 0.5, respectively. These $\beta$ values correspond to failure probabilities of 1.27% and 5.92%. The mean-value point ($\mu_Q, \mu_Q$) is the most likely event and in the safe domain, but the spring can still rupture when $Q_s$ decreases and $Q$ increases to a common value (60 kN for Case A, and 80.488 kN for Case B). The chances of $Q_s$ and $Q$ attaining values such that $Q > Q_s$ (thereby leading to spring rupture) are 1.27% for Case A and 5.92% for Case B. Case B is less safe because of its higher uncertainty in the applied load $Q$.

One can seek the required mean rupture strength $Q_r$ (with $\sigma = 0.2 \mu$ as in Fig. 1) to achieve a target reliability index of 2.5, say. The results are shown in Fig. 3.

![Figure 2. Random numbers generated during Monte Carlo simulations need to obey the physical requirement that the water depth $z_s$ in the tension crack cannot be greater than the depth $z$ of the tension crack.](image2)

![Figure 3. The resistance and load factors $x_i^*/\mu$ are corollaries of reliability-based design and reflect uncertainties & sensitivities.](image3)
In a reliability-based design (such as the case in Figure 3) one does not prescribe the ratios \( x/\text{mean} \), but leave it to the expanding dispersion ellipsoid (Fig. 4) to seek the most probable failure point (i.e. the design point) on the limit state surface, a process which automatically reflects the uncertainties, sensitivities and correlations of the parameters in a way that code-specified partial factors cannot. Besides, one can associate a probability of failure for each target reliability index value. The abilities to (i) seek the most-probable design point without presuming any code-specified load and resistance factors (or partial factors) and (ii) automatically reflect sensitivities from case to case are merits of the reliability-based design approach.

3.2 Positive reliability index only if the mean-value point is in the safe domain

In reliability analysis and reliability-based design one needs to distinguish negative from positive reliability index. The computed \( \beta \) index can be regarded as positive only if the performance function value is positive at the mean value point. Although the discussions in the next paragraph assume normally distributed random variables, they are equally valid for the equivalent normals of nonnormal random variables in FORM.

The five random variables of the rock slope in Fig. 2 include the shear strength parameters \( c \) and \( \phi \) of the discontinuity plane which is inclined at \( \varphi_p \). In the 2-dimensional schematic illustration of Fig. 4, the limit state surface (LSS) is defined by performance function \( g(x) = 0 \). The safe domain is where \( g(x) > 0 \), and the unsafe domain where \( g(x) < 0 \). The mean-value point of Case I is in the safe domain and at the centre of a standard-deviation deviation dispersion ellipse, or ellipsoid in higher dimensions. As the dispersion ellipsoid expands, the probability density on its surface diminishes. The first point of contact with the LSS is the most-probable failure point, also called the design point. The reliability index \( \beta \) of Case I is therefore positive and represents the distance (in units of directional standard deviations) from the safe mean-value point to the unsafe boundary (the LSS) in the space of the random variables. The corresponding probability of failure \( \Phi(-\beta) \) is less than 0.5. If the mean-value point sits right on the LSS, the probability of failure is 0.5 because \( \beta = 0 \) when the \( \beta \)-ellipsoid reduces to a point on the LSS. In contrast, Case II’s mean-value point is already in the unsafe domain, and the computed \( \beta \) must be given a negative sign because it is the distance from the unsafe mean-value point to the safe boundary (the LSS). Case II’s probability of failure is greater than 0.5.

The reinforcing force \( T \) required to achieve a target reliability index \( \beta \) (e.g. 2.5) for the slope of Fig. 2 can be obtained as follows:

(i) If the performance function \( g(x) \) is positive at the mean-value point (as in Case I), \( \beta \) is positive. Perform FORM reliability analysis with increasing \( T \) until \( \beta = \beta_{\text{target}} \).

(ii) If the performance function \( g(x) \) is negative at the mean-value point (as in Case II), assign a negative sign to the computed \( \beta \). Perform FORM reliability analysis with increasing \( T \) until \( \beta = -\beta_{\text{target}} \). In this case \( \beta \) will move from negative value to zero and then to \( \beta_{\text{target}} \).

The reliability-based design of the embedment depth of an anchored sheet pile wall in Low (2005) provides another example of the need to distinguish negative \( \beta \) from positive \( \beta \) values.

4. SLOPE RELIABILITY ANALYSIS ACCOUNTING FOR SPATIAL VARIATION

The geological processes of soil formation impart spatial auto-correlation to most soil properties. Two methods used by the writer in his reliability analysis accounting for 1-D spatial variability are: (i) the method of interpolated autocorrelations, and (ii) the method of autocorrelated slices. These are described below. It is natural and straightforward to extend both methods to 2-D spatial variability in limit equilibrium analysis. The method of interpolated auto-correlations can also be applied to 2-D finite element method.

4.1 Method of interpolated autocorrelations

This method was used in Low and Tang (1997), where the vertical spatial variation of the \( c_u \) of the soft clay beneath an embankment was modeled by six autocorrelated \( c_u \) random variables, and used to interpolate for the \( c_u \) at 20 points along the slip surface. Low (2001) investigated the effects of discretized one dimensional random process on the computed reliability index, and found that the computed \( \beta \) index was practically the same regardless of the number of discrete points used to represent the one-dimensional \( c_u \) random field, provided the spacing between the discrete \( c_u \) points is smaller than the autocorrelation distance. This is consistent with the suggestion of Vanmarcke (1980) that no reduction on standard deviation is required when the spacing is less than the scale of fluctuation (which is two times the autocorrelation distance if autocorrelation is modeled by the exponential model).

4.2 Method of autocorrelated slices

This method was used in Low et al. (2007)’s limit equilibrium analysis. The horizontal variation of both \( c_u \) and unit weight \( \gamma \) of a clay slope were modelled. Since there were 24 spatially autocorrelated slices, the size of the correlation matrix was 48x48. The design point obtained represents the most probable combination of the 24 values of \( c_u \) and the 24 values of \( \gamma \) which would cause failure. An interesting finding is that most of the design point values of \( \gamma \) are above their mean values, as expected for loading parameters, but, somewhat paradoxically, there are some design-point values of \( \gamma \) near the toe which are below their mean values. The implication is that the slope is less safe when the unit weights near the toes are lower. This implication has been verified by deterministic runs using higher \( \gamma \) values near the toe, with resulting higher factors of safety. It would be difficult for design code committee to recommend partial factors such that the design values of \( \gamma \) are above the mean along some portions of the slip surface and below their mean along other portions. In contrast, the design point is located automatically in FORM analysis, and reflects sensitivity and the underlying statistical assumptions from case to case in a way specified partial factors cannot.

![Figure 4](image-url)  
Figure 4. Increasing the magnitude of reinforcing force \( T \) will increase the \( \beta \) value of Case II from negative to zero to target \( \beta \).

![Figure 5](image-url)  
Figure 5. Modelling spatial variability using the method of autocorrelated slices.
5. FORM AND BIMODAL BOUNDS FOR SYSTEM FAILURE PROBABILITY OF A SOIL SLOPE

Established equations for bimodal bounds of probability of system failure are described in Ang & Tang (1984), Melchers (1999) and Haldar & Mahadevan (1999), for example. Low et al. (2009) presented a practical computational approach for automatic and efficient implementation of the Kounias-Ditlevsen bimodal bounds for systems with multiple failure modes, in which occurrence of one or more failure modes constitutes system failure. For example, a semi-gravity retaining wall could fail by sliding, overturning, or bearing capacity failure. For a slope with two different soil layers, failure can occur due to a slip surface entirely in the upper soil layer only or one passing through both the upper and lower soil layers, for example. The Kounias-Ditlevsen bimodal bounds estimate the probability of system failure based on knowledge of single-mode failure probabilities and correlations among the failure modes.

The slope in two clayey soil layers of Fig. 6 was analyzed by Ching et al. (2009) using Monte Carlo simulation (MCS) and importance sampling (IS) methods. The upper clay layer is 18 m thick, with undrained shear strength $c_{u1}$; the lower clay layer is 10 m thick, with undrained shear strength $c_{u2}$. The undrained shear strengths are lognormally distributed and independent. A hard layer exists below the second clay layer. Since the shear strengths are characterized by $c_{u1}$ and $c_{u2}$, with $\phi_1 = 0$, Bishop’s simplified method and the ordinary method of slices yield the same results, and either can be used.

In this case where the upper clay layer is weaker than the lower clay layer, deterministic analysis will indicate two local minimums of the factor of safety, corresponding to two critical slip circles, one entirely in the upper clay layer and the other passing through both layers. Likewise, one can locate two reliability-based critical slip circles. As shown in Low et al. (2009), the FORM reliability indices for the two modes are 2.795 and 2.893, respectively, and the Kounias-Ditlevsen bimodal bounds on system failure probability are 0.432% and $P_{F, sys} < 0.441%$. (A separate 8-mode analysis, in which the six additional modes are not local minimums, yielded the same six bounds of system failure probability for the two-layered soil slope.)

For comparison, Ching et al. (2009) reported a failure probability of range 0.37%-0.506% (from Monte Carlo mean $P_{F, sys}$ of 0.44% and c.o.v. of 15.04%).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{u1}$ Lognormal</td>
<td>120 kPa</td>
<td>36 kPa</td>
</tr>
<tr>
<td>$c_{u2}$ Lognormal</td>
<td>160 kPa</td>
<td>48 kPa</td>
</tr>
</tbody>
</table>

Figure 6. The system failure probability obtained from FORM and Kounias-Ditlevsen bounds agrees with MC simulations.

The above example has two failure modes which are local minimums and almost equally likely to occur. Provided one accounts for multiple failure modes, the computed system failure probabilities are practically identical whether one uses FORM or Monte Carlo simulations. This writer always finds comparative studies by various approaches (e.g. Monte Carlo simulations, importance sampling, FORM with system bounds) beneficial and interesting, and think that the various approaches can play complementary (not antagonistic or exclusive) roles.

6. SUMMARY AND CONCLUSIONS

All methods of engineering and scientific analysis have strengths and weaknesses, but all can contribute by playing complementary roles. This paper presents the author’s experience of some potential pitfalls in Monte Carlo simulations (a very valuable method nonetheless, and used by the author frequently for verifications and comparative studies), and some merits and subtleties of FORM reliability analysis and reliability-based design. Two methods of accounting for spatial autocorrelations used by the author are also described, namely the method of interpolated autocorrelations and the method of autocorrelated slices. The paper ends with a favourable comparison of FORM (plus Kounias-Ditlevsen bounds) with a reported case of Monte Carlo simulations for a slope with two clay layers.

7. REFERENCES


