Identification and Robust Tracking Control of a Single-phase Rotary Motor with Halbach Permanent Magnet Array Design*

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Abstract—The single-phase electromagnetic motor with the circular permanent magnet Halbach array design, has advantage of large force constant compared with its 3-phase counterpart, if the rotation is within 60 degrees range from its neutral position. However, to authors’ best knowledge, no existing literature mentions on tracking control application using such a single phase rotary motor, probably due to its angular dependent force sensitivity. This paper first models a typical two-pole single phase rotary motor with Halbach circular array, with consideration of back-EMF, friction torque and position dependent characteristic. Followed by this, a dual-relay configuration is applied to closed-loop identification of model-parameters by limit cycle experiments. Later, a composite controller, composed of a model-based controller, a time-delay controller and a sliding mode controller, is capable to achieve the high-speed tracking of reference trajectory with presentation of model uncertainty and disturbances. Simulation based on the actual design parameters shows the practical appeal of the proposed approach.

I. INTRODUCTION

The single-phase electromagnetic (EM) actuators, compared with its three-phase counterparts, requires no commutation in the current-loop, which provides high-bandwidth and simple force-position relationship [1]. The linear version of it, named the voice coil, is capable to provide typically few millimeters travel stroke with nano-positioning accuracy with flexure design [2], [3]. In recent years, the Halbach array becomes a popular components in the design of EM actuators, for providing the large magnetic field towards side of the moving coil while minimize the flux in the opposite side. Especially, the 2-pole circular Halbach array, which provides a ideal uniform field inside the cylinder, is applied to design the single-phase rotary motor, as shown in Figure 1. Remarkably, it provides larger force with same length of coil and same amperes of command current within its travel range $\theta < 60^\circ$, compared with the conventional three-phase permanent magnet synchronous motor (PMSM), as proved in the appendix. In 2009, An integral linear-rotary actuator for high-speed pick-and-place application is developed by SIMTech [4], where the design of such a single-phase rotary motor is adopted in the rotary module [5], as shown in Figure 2. However, to authors’ best knowledge, no current literature is working on using such a motor for high-speed tracking application, possibly due to its position dependent force sensitivity. The identification method for such input-output coupled nonlinear system is not available till now either.

Relay is probably the simplest and strongest nonlinearity, which has been used for auto-tuning of proportional-integral-derivative (PID) controller since 1980s. Different forms of relay structures, such as hysteretic relay [6], relay with delay [7] and dual-relay [8] are proposed. Since 1990s, the application of relay feedback tends to identified the compounded linear and nonlinear model parameters in motion systems by tuning of relay parameters to excite oscillations, such as friction [9] and force ripples [10]. Although time-domain based method are occasionally used for system identification based on solving of a series of step-response equation, the application is applicable on identify the simple form of nonlinearity, such as Coulomb friction [11]. For more complicated form of nonlinearity, existing literatures on relay feedback identification only focus on the systems where linear and nonlinear portions can be separated accordingly, where the harmonics balanced method are applicable [12]. Due to the approximation characteristic of describing function, the error of obtained parameter will be around 10-30%, therefore a following-up design of adaptive or robust controllers to cater such modeling error is required for fine tracking application [13].

In this paper, a typical single-phase EM with circular Halbach array is modeled first, taking in consideration of the back-EMF, friction torque, and disturbance. From the model, it is clearly shown that the motor presents a nonlinear dynamics which is highly dependent on the instant position. In the following part, we show that, although the position dependent nonlinear terms exists on the remained subsystems.
Fig. 1. Single-phase rotary motor with Halbach permanent magnet design.

Fig. 2. A 2-DOF linear rotary actuator for high-speed pick-and-place application.

Fig. 3. Force generated in single-phase permanent magnet rotary voice coil with Halbach design.

Besides the Coulomb friction, we are still capable to identify the model parameters at different equilibrium points with excitation of appropriate oscillations using a dual-relay feedback configuration. In the next part, a composite controller is proposed for robust high-speed tracking application. The controller composes of a model-based controller, taking in the identified parameter from relay-experiment, a time-delay compensator to cancel the slow varying disturbance, and a sliding mode controller to make the tracking error converge to zero. Simulation based on the actual motor design parameters shows the practical appeal of the proposed methods.

II. Modeling and Identification of the Rotary Voice Coil with Dual-Relay Feedback

Figure 3 illustrates the effective force and velocity components for producing torque and rotating the coil. The driving torque of the coil

\[ T_d = Fr \cos \theta = nB lir \cos \theta \]

leads to

\[ K_d \cos \theta i. \]  

where \( n \) is the number of turns, \( B \) is uniformly flux density by rotary Halbach array, \( l \) is effective length per turn. \( r \) is radius of rotation. Mechanically,

\[ T_d = J \ddot{\theta} + T_f + T_e + T_u. \]  

where

\[ T_f = k_c \text{sgn} (\dot{\theta}), \]

is friction torque, and

\[ T_e = \frac{n^2 B^2 l^2}{R} \cos^2 \theta \dot{\theta}, \]

\[ = K_e \cos^2 \theta \dot{\theta}. \]  

is the torque from back-EMF, and \( R \) is coil resistance and \( T_u \) is the residue torque. Substitute (1), (3) and (4) into (2),

\[ K_d \cos \theta i = J \ddot{\theta} + K_c \text{sgn}(\dot{\theta}) + K_e \cos^2 \theta \dot{\theta} + T_u, \]

or equivalently

\[ a \ddot{\theta} + b \cos^2 \theta \dot{\theta} = \cos \theta i - c \text{sgn}(\dot{\theta}) - d, \]  

where \( a = J / K_d, b = K_e / K_d, c = K_c / K_d \) and \( d = T_u / K_d \) are all position independent.

In (6), define the position dependent parameters \( \tilde{a}(\theta) = a / \cos \theta, \tilde{b}(\theta) = b \cos \theta, \tilde{c}(\theta) = c / \cos \theta, \tilde{d}(\theta) = d / \cos \theta, \) and let \( \tilde{i} = i - \tilde{c} \text{sgn}(\dot{\theta}). \) The dynamical equation (6) is now expressed as

\[ \tilde{a} \ddot{\theta} + \tilde{b} \dot{\theta} = \tilde{i} - \tilde{d}, \]  

Let us ignore \( \tilde{d} \) in the identification stage, the transfer function \( G(s) \) of linearized model (7) from \( i \) to \( \theta \), around the equilibrium point \( [\theta, \dot{\theta}] = [\vartheta, 0] \), is given by

\[ G(s) = \frac{\theta(s)}{i(s)} = \frac{1}{s[\tilde{a}(\theta)s + \tilde{b}(\theta)]}. \]  

A typical dual-relay feedback is introduced to excite limit cycles around \( \theta = \vartheta \).

\[ i = \alpha \text{sgn}(\vartheta - \theta) + \beta \int_0^t (\text{sgn}(\vartheta - \theta))dt, \]  

where \( \alpha \) and \( \beta \) are proportional and integral gains of the dual-relay accordingly. Figure 4 shows overview of this voice coil model under dual-relay feedback. Notice that (7) contains
position dependent parameters, to improve the accuracy of identification, by tuning $\alpha$ and $\beta$, it is possible to let $A < |\theta - \hat{\theta}|$ [12], [11], so that

$$|\cos \theta - \cos \hat{\theta}| < \delta \cos \theta,$$  \hfill (10)

where $0 < \delta < 1$. The allowable maximum amplitude is the lower bound of this two curve. As $\theta$ angle increases, the force constant is more sensitive with respect to variation angle. Thus, smaller amplitude of oscillation is allowed at larger $\theta$. For example, when relay experiment is carried at $\theta = 30^\circ$, only $A < 1^\circ$ is allowed for 1% variation in $\cos \theta$, compared with the allowable $A < 8^\circ$ when $\theta = 0^\circ$. Remarkably, it is harder to tune the dual-relay to achieve such small amplitude of oscillation while maintaining adequate small oscillation frequency. Furthermore, the limited rotary encoder resolution will affect the accuracy of experiment during extra-small amplitude oscillation.

The describing function of nonlinear components, including the dual-relay and the Coulomb friction is

$$N(A) = \frac{4\alpha}{\pi A} - \frac{4A(\beta - \hat{c})}{\pi A},$$  \hfill (11)

Using the harmonics balance condition $N(A) = -1/G(j\omega)$, and excite the oscillation around the equilibrium point $[\theta, 0]^T$ with frequency $\omega$ and sufficient small amplitude $A$ satisfying condition (10), yielding

$$\hat{a} = \frac{1}{2} \left( \frac{4\alpha_1}{\pi A_1^2} + \frac{4\alpha_2}{\pi A_2^2} \right) = \hat{a} / \cos \theta,$$  \hfill (12)

$$\hat{b} = \frac{4(\beta_2 - \hat{\beta}_1)}{\pi(A_2^2 - A_1^2)} = \hat{b} \cos \theta,$$  \hfill (13)

$$\hat{c} = \frac{\beta_1 A_2^2 - \beta_2 A_1^2}{A_2^2 A_1^2} = \hat{c} / \cos \theta$$  \hfill (14)

where $\hat{\cdot}$ denotes the estimation of $\cdot$, and $(\cdot)_i$ is the value of $\cdot$ in the $i$th experiment accordingly.

With dual-relay excitation of limit cycles at equilibrium point $\theta = \hat{\theta}$, the parameters of $a$, $b$ and $c$ are simply identified by (12)–(14).

III. CONTROLLER DESIGN

As the conventional motion control system, a composite controller with model-based “feedforward” and feedback is designed for tracking of a trajectory $(\hat{\theta}_d, \hat{\dot{\theta}}_d, \hat{\ddot{\theta}}_d)$, with identified $\hat{a}$, $\hat{b}$, $\hat{c}$. Set $\bar{i} = i \cos \theta$, then $\bar{i} = i_m + i_d + i_s$, where $i_m$ is the model-based controller, $i_d$ is the time-delay compensator for cancelation of slow-varying disturbances, $i_s$ is a sliding mode controller for cancelation of residue uncertainty for perfect tracking.

The model-based controller is designed based on the relay identification results in previous section as

$$i_m = \hat{a}\ddot{\theta} + \hat{b}\cos^2 \theta \dot{\theta} + \hat{c} \text{sgn}(\theta_d),$$  \hfill (15)

where the term of $\hat{b}\cos^2 \theta$ contains the sensor feedback information. Set $\hat{\theta} = \theta_d - \hat{\theta}$, $\hat{\dot{\theta}} = \theta_d - \dot{\theta}$, and $\theta = \theta_d - \hat{\dot{\theta}}$.

Substitute (15) into (6), we have

$$\ddot{\hat{\theta}} + \hat{b}\cos^2 \theta \dot{\hat{\theta}} = (a - \hat{a})\ddot{\theta} + (b - \hat{b})\cos^2 \theta \dot{\theta} + c \text{sgn}(\hat{\theta}_d) - \hat{c} \text{sgn}(\theta_d) + d - i_d - i_s.$$  \hfill (16)

Define $\delta = (a - \hat{a})\ddot{\theta} + (b - \hat{b})\cos^2 \theta \dot{\theta} + c \text{sgn}(\hat{\theta}_d) - \hat{c} \text{sgn}(\theta_d) + d$. And let $i_d = \delta(t - \tau)$ to cancel the slow-varying disturbances.

$$i_d = \delta(t - \tau) = (a - \hat{a})\ddot{\theta}(t - \tau) + (b - \hat{b})\cos^2 \theta(t - \tau)\dot{\theta}(t - \tau) + c \text{sgn}(\hat{\theta}_d(t - \tau)) - \hat{c} \text{sgn}(\hat{\theta}_d(t - \tau)) + d(t - \tau).$$

Define the sliding surface $\sigma = \lambda \hat{\theta} + \dot{\hat{\theta}}$. Consider a Lyapunov function candidate $V = \dot{\sigma}^2/2$

$$\dot{V} = \ddot{\sigma}\dot{\sigma} = \ddot{\sigma}(\lambda \hat{\theta} + \dot{\hat{\theta}}) = \sigma[\ddot{\sigma}\hat{\theta} + \hat{\theta} - \ddot{\theta} - \dot{\theta}] - \ddot{\sigma}^2/2 - \dot{\sigma}.$$  \hfill (19)

If

$$|\delta(t) - \delta(t - \tau)| < \epsilon,$$  \hfill (20)

then let

$$i_s = (\hat{a}\lambda - \hat{b}\cos^2 \theta)\dot{\theta} + \epsilon \text{sgn}(\sigma) + k\sigma,$$

$$= k\lambda\dot{\theta} + (\hat{a}\lambda - \hat{b}\cos^2 \theta)\dot{\theta} + \epsilon \text{sgn}(\sigma).$$  \hfill (21)
TABLE I
SIMULATION PARAMETERS SETTING.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value (Unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>0.2 kg</td>
</tr>
<tr>
<td>r</td>
<td>0.02 m</td>
</tr>
<tr>
<td>R</td>
<td>3 Ω</td>
</tr>
<tr>
<td>K_d</td>
<td>0.03 N.m/A</td>
</tr>
<tr>
<td>K_c</td>
<td>1 × 10⁻³ N.m</td>
</tr>
<tr>
<td>T_u</td>
<td>1 × 10⁻⁵ sin 0.1t</td>
</tr>
</tbody>
</table>

We have $\dot{V} \leq -k \sigma^2$. Thus, $\sigma \to 0$, then $\dot{\theta} \to 0$.

In summary, with estimation of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, the following controller $i(t)$, ensure tracking of arbitrary smooth reference signal $(\theta_d, \dot{\theta}_d, \ddot{\theta}_d)$.

$$i(t) = \frac{i(t)}{\cos \theta(t)}, \quad \text{(22)}$$

$$\ddot{i}(t) = i_m(t) + i_d(t) + i_s(t), \quad \text{(23)}$$

$$i_m(t) = \tilde{a}\dot{\theta}_d + (\tilde{b}\cos^2 \theta(t))\dot{\theta}_d + \tilde{c} \text{sgn}(\dot{\theta}_d), \quad \text{(24)}$$

$$i_d(t) = \ddot{i}(t - \tau) - \tilde{a}\dot{\theta}(t - \tau) - \tilde{b}\cos^2 \theta(t - \tau)\dot{\theta}(t - \tau) - \tilde{c} \text{sgn}(\dot{\theta}(t - \tau)), \quad \text{(25)}$$

$$i_s(t) = k_{\lambda}\ddot{\theta} + (k + \tilde{a}\lambda - \tilde{b}\cos^2 \theta)\dot{\theta} + \varepsilon \text{sgn}(\sigma), \quad \text{(26)}$$

Remark 3.1: By using the controller (23)-(26), there are four parameters needed tuning, that is, $k$, $\lambda$, and $\varepsilon$ in $i_s(t)$, and $\tau$ in $i_d(t)$.

Remark 3.2: In practice, to avoid the excess chattering, the $\text{sgn}(\sigma)$ function in (26) is replaced by $\text{sat}(\sigma/\eta)$, where $\eta$ is a small positive number.

IV. SIMULATION

A. Parameter identification via relay-feedback

In the following part of simulation, the primary parameters are set according to the motor design as in Table I. From here, the secondary parameters are derived as $J = mr^2/2 = 4 \times 10^{-5}$, $K_s = K_d^2/R = 3 \times 10^{-4}$. Thus, the system parameters $a = J/K_d = 0.00133$, $b = K_c/K_d = 0.01$, $c = K_c/K_d = 0.0333$, $d = T_u/K_d = 3.333 \times 10^{-4} \sin 0.1t$.

Now, the relay experiments are conducted at the equilibrium point $\theta = 0$ with data sampling rate of 100μs for model parameter identification. Two sets of gains of relay and the corresponding limit cycle amplitudes and frequencies are listed in Table II and Figure 5. Notice that the both oscillations are within 8° or 0.14rad/s to ensure small effects on the nonlinear position dependent terms. Thus, by (12)–(14), the parameters are identified as summary in Table III. The maximum percentage error being identified is merely 6.94% (parameter b).

B. Robust tracking control

For tracking control, a 4-th order reference profile with information of angular position, velocity, acceleration, jerk and snap is designed [14], as specified in Table IV, for repetitive motion between $-45^\circ$ and $45^\circ$. For comparison of the control scheme (22)-(26), the following composite controller (27)–(30) are also tested, consisting of the same model-based control $i_m$ as in (24) and PID feedback control $i_b$.

$$i = \frac{\ddot{i}}{\cos \theta}, \quad \text{(27)}$$

$$\ddot{i} = i_m + i_b, \quad \text{(28)}$$

$$i_m = \tilde{a}\dot{\theta}_d + \tilde{b}\cos^2 \theta \dot{\theta}_d + \tilde{c} \text{sgn}(\dot{\theta}_d), \quad \text{(29)}$$

$$i_b = k_p\ddot{\theta} + k_i \int_0^t \dot{\theta} \, dt + k_d \dot{\theta}, \quad \text{(30)}$$
TABLE V
PARAMETERS OF TWO CONTROLLERS.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Controller I Value</th>
<th>Controller II Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>$10^{-4}$</td>
<td>$k_p$ 0.3</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$10^{-5}$</td>
<td>$k_i$ 0.05</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.01</td>
<td>$\varepsilon$ 0.3</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.03</td>
<td>$\tau$ 0.3</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$10^{-5}$</td>
<td>$\eta$ 0.3</td>
</tr>
</tbody>
</table>

For the test of robustness of proposed control schemes, the model parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ are selected using estimated value of Table III for both Controller I and II, while the parameters of the motion system still follows Table I. The Controller II is fine tuned in such a way that the control signals has similar amplitude as Controller I for a fair comparison, and the overshoot is minimized partially due to the 4-th order smooth trajectory as in Table IV. The parameters of the proposed controller (Controller I) and the composite controller (Controller II) are listed in Table V. From the solid line of Figure 6, we can see that maximum tracking error by using Controller I is $1.67 \times 10^{-3}$rad, compared with the $1.732 \times 10^{-3}$rad by using Controller II, which is almost identical, thanks to the small identification errors by dual-relay in this simulation example. However, the error of model parameters identified in relay experiment can be as large as 10-30%. From simulation, we find that if the modeling error increases to 10%, the maximum error by Controller I and II are $2.6 \times 10^{-3}$rad and $8.4 \times 10^{-3}$rad accordingly, where the error of Controller II is 2.23 times larger than that in Controller I. More extremely, when the modeling error increases to 30%, the maximum tracking error by Controller I is still merely $3.3 \times 10^{-4}$rad, while this shoots up to $2.7 \times 10^{-2}$rad by using Controller II. In other words, Controller I has more robust performance in case of modeling error. The RMS values of control signals in both controllers are similar but Controller I has more chattering in switching around sliding surface to cancel the uncertainty.

In simulation, we would like to also investigate effects of the parameters $\tau$, $k$, $\lambda$ and $\varepsilon$ in Controller I to the tracking performance, when the identification error is 30%. Figure 7 shows the tracking performance when varying a single parameter while keeping other parameters as default values as in Table V. From here, it is concluded that the time delay term $\tau$ is a key parameter which affects the tracking performance, when $\tau$ is too large, it is hard to guarantee (20) due to the disturbances. Meanwhile, it is good to increase $\varepsilon$ so that (20) is easily held. $\lambda$ and $k$ work together to determine the convergent speed of the error to the sliding surface.

Next, we would also like to investigate the tracking performance proposed Controller I with additional strong disturbance on the fly. A strong disturbance with frequency 0.3rad/sec and amplitude 20 and 100 times of original disturbance are injected to the system between 2 and 3 second of motion. Figure 8 shows the tracking performance of such an evaluation. From this figure, it can be concluded that the tracking performance will be still maintain to a reasonable range even with large disturbance is presented, and the tracking performance will be gradually restore to the original range after the strong disturbance is removed.

V. CONCLUSION

The single-phase rotary servo motor with PM Halbach array design has large driving torque. However, due to its nonlinear torque constant, back-EMF force and friction in the bearing, there is no existing literature on using it for
high-speed tracking application. In this paper, the dynamics of this type of servo motor is modeled with consideration of above nonlinearities. A dual-relay feedback is configured to closed-loop identify the model parameters with excitation of suitable steady oscillation. Later, a composite model-based controller with time-delay and sliding mode control modules is proposed to track the typical motion profiles with parameter error and external disturbances. Simulation work is performed to verify the effectiveness of proposed methods.

APPENDIX

Consider an \( n \)-turn coil with effective length \( L \) per turn, radius \( r \) with constant command current \( i \), cutting the magnetic field with uniform flux density \( B \) from circular Halbach array. Since it is single phase, ignore the dynamics of inductance, the actual current equals to the command current. Thus, the torque \( T_1 \) generated by this single phase motor is

\[
T_1 = nBlr \cos \theta_i.
\]

Now, as shown in Figure 9, if the same length of coil is equally distributed to three phase permanent magnet synchronous motor (PMSM) with same radius and magnetic flux density, as shown in Figure 3. Under sinusoidal commutation with \( dq0 \) transformation, when constant command current \( i = I_q \) is given, the overall torque produced by three phases are

\[
T_3 = \frac{3}{2} nBlri = \frac{1}{2} nBlri.
\]

Thus, when \( |\theta| < 60^\circ \), \( T_1 > T_3 \). In other words, for same total effective length of coil, and same command current, the single phase motor with Halbach design in Figure 3 produces larger force than 3-phase PMSM when rotation angle is less than 60° from the neutral position.

REFERENCES