Modeling and Robust Output feedback Tracking Control of a Single-phase Rotary Motor with Cylindrical Halbach Array

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Abstract—The single-phase electromagnetic motor with a circular Halbach array design offers the advantage of generating a large torque. However, to the authors’ best knowledge, no existing literature has presented the tracking control of this motor. This is probably due to its DC gain and time constants that are rotary-position-varying. This paper first models a typical two-pole single phase rotary motor based on a Halbach circular array with considerations of its back-EMF, frictional torques and position dependent characteristics. Next, a dual-relay configuration is applied in a closed-loop to identify the model parameters through limit cycle experiments. Next, a composite controller that composes of a model-based controller, a time-delay compensator and a sliding mode controller, was proposed with the capability to achieve the high-speed tracking of a reference trajectory in the presence of model uncertainty and external disturbances. A non-linear high gain observer was designed to estimate unknown states to allow output feedback realization. An experiment was carried out to ascertain the performance of the proposed composite controller and to benchmark it to a PID controller with a model-based feedforward element. The proposed controller presents a noticeable improvement and offers a clearly viable alternative to the PID controller for this particular application.

I. INTRODUCTION

There is much commercial and research interest in the development of advanced pick and place mechanisms (PPM). Currently, conventional PPMs mainly make use of permanent magnet synchronous motors (PMSMs) which have been rigorously studied [1]. In recent years, the Halbach array configuration has gained popularity and traction amongst developers and researchers. One of the most advantageous features of the Halbach arrangement in the 2 pole configuration is that it provides a large magnetic field towards the side of the moving coil while minimizing the flux on the opposite side. As a result of this concentration of flux, the torque of the Halbach arrangement is greater than that of a 3 phase PMSM within a range of $|\theta| < 60^\circ$ as shown in the subsequent paragraph.

Consider an $n$-turn coil with effective length $l$ per turn, radius $r$ with constant command current $i$, cutting the magnetic field with uniform flux density $B$ from circular Halbach array. Since it is single phase, the dynamics of the inductance can be ignored. Hence, the actual current equals to the command current. Therefore, the force generated by this single phase motor is

$$T_1 = nBlr \cos \theta i.$$

Now, as shown in Figure 2, if the same length of coil is equally distributed into a three phase permanent magnet synchronous motor (PMSM) with same radius and magnetic flux density, as shown in Figure 3. Under sinusoidal commutation with dq0 transformation and when constant command current $i = I_d$ is given, the overall torque produced by the three phase PMSM are given as,

$$T_3 = \frac{3}{2} nBl^3 r i = \frac{1}{2} nBlr i.$$

Thus, when $|\theta| < 60^\circ$, $T_1 > T_3$. In other words, for same total effective length of coil, rotor radius and command current, the single phase motor with Halbach design in Figure 3 produces a larger torque than 3-phase PMSM when its rotation angle is less than $60^\circ$ from the neutral position.

In addition to the increase in the flux over this range, the single phase Halbach motor requires no commutation in the current loop. This gives rise to higher bandwidth and a simple force-position relation [2]. Lastly, in the ideal implementation of the 2 pole Halbach array, the magnetic field is uniform within the stator which is shown in Figure 1.

To authors’ best knowledge, no prior literature has presented a tracking control application using a 2 pole, single phase Halbach rotary motor. This is probably due to the varying DC gain and time constants which are closely related to the rotor’s angular position. Therefore, the aim of this paper is to present a mathematical model of the said motor, a method of estimating its parameters and a composite-sliding mode output feedback controller which offers a tracking performance that supersedes standard proportional-integral-derivative controllers and its variants.

This paper first outlines the mathematical model of a single phase circular Halbach array. This model takes into account the back-EMF, frictional torque and disturbance. It is evident that the motor’s nonlinear dynamics is highly dependent on its instantaneous angular position. This poses a challenge to identification as there are position dependent nonlinear terms that exist other than Coulomb friction.

The paper explores through dual-relay feedback, model parameters estimation at different equilibrium points through an appropriate excitation. Relay identification is a process

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Fig. 1. Single-phase rotary motor with Halbach permanent magnet design.

Fig. 2. Comparison of single-phase rotary motor and 3-phase PMSM.

Fig. 3. Force generated in single-phase permanent magnet rotary voice coil with Halbach design.

whereby the plant is excited by self-generated oscillations in order to determine the parameters that make up its mathematical model. Extensive studies have been conducted since 1980s where several relay structures have been proposed such as hysteretic relay [3], relay with delay [4] and dual-relay [5]. However, there exist 10-30% error in identification through relay feedback. However, this method is chosen as the approach is able to detect nonlinearities such as Coulomb friction efficiently [6]. Therefore, it is imperative that a robust control scheme is used to overcome modeling errors and uncertainties. Hence, in the later portion of this paper, a composite controller will be presented.

This composite controller comprises of 3 individual controllers which are the model-based controller, time-delay compensator and a sliding mode controller. Each individual controller fulfills a specific task; the model-based controller helps to give the controller advanced plant information, the time-delay compensator helps to eliminate slow varying disturbances and the sliding mode controller aims to eliminate tracking errors. In order to make this proposed controller practically realizable and appealing, a high gain observer is also presented to allow for output feedback implementation.

II. MODELING AND IDENTIFICATION OF THE ROTARY VOICE COIL WITH DUAL-RELAY FEEDBACK

The effective force and velocity components for producing torque and rotating the rotor are illustrated in Figure 3. The driving torque of the coil, 

\[ T_d = F r \cos \theta = n B l r \cos \theta i \]

\[ = K_d \cos \theta i. \]  

where \( F \) is the tangential force acting on the rotor, \( r \) the radius of rotation, \( \theta \) the angle of rotation, \( n \) the number of turns, \( B \) the uniform flux density of rotary Halbach array, \( l \) the effective length per turn and \( i \) the command current input. The dynamics of the system given by,

\[ T_d = J \ddot{\theta} + T_f + T_e + T_u. \]  

where \( T_f = k_c \text{sgn}(\dot{\theta}), \) 

is the frictional torque and 

\[ T_e = \frac{n^2 B^2 l^2 r^2 \cos^2 \theta \dot{\theta}}{R} \]

\[ = K_e \cos^2 \theta \dot{\theta}. \]

is the impending torque from back-EMF. \( J \) the moment of inertia, \( R \) the coil resistance and \( T_u \) is the residual torque.

Substitute (1), (3) and (4) into (2),

\[ K_d \cos \theta i = J \ddot{\theta} + K_e \text{sgn}(\dot{\theta}) + K_e \cos^2 \theta \dot{\theta} + T_u, \]

or equivalently

\[ a \ddot{\theta} + b \cos^2 \theta \dot{\theta} = \cos \theta i - c \text{sgn}(\dot{\theta}) - d, \]

where \( a = J / K_d, b = K_e / K_d, c = K_e / K_d \) and \( d = T_u / K_d \) are all position independent. The position dependent parameters which mirror parameters found in (6) can be defined as \( \tilde{a}(\theta) = a / \cos \theta, \tilde{b}(\theta) = b \cos \theta, \tilde{c}(\theta) = c / \cos \theta, \)

\( \ddot{\theta}(\theta) = \ddot{\theta} / \cos \theta, \) and let \( i = i - \tilde{c} \text{sgn}(\dot{\theta}). \) By applying
the position dependent parameters into equation (6) can be expressed as,

\[ \bar{a}\dot{\theta} + \bar{b}\dot{\theta} = \dot{i} - \bar{d}, \]

(7)

The disturbance \( \bar{d} \) is ignored in the identification stage. The transfer function \( G(s) \) of linearized model (7) from \( \dot{i} \) to \( \theta \), around the equilibrium point \( [\theta, \dot{\theta}] = [0, 0] \) is given by,

\[ G(s) = \frac{\theta(s)}{\dot{i}(s)} = \frac{1}{s[\bar{a}(\theta)s + \bar{b}(\theta)]}. \]

(8)

A typical dual-relay feedback can be introduced to excite limit cycles around \( \theta = \vartheta \),

\[ i = \alpha \text{sgn}(\theta - \vartheta) + \beta \int_0^t (\text{sgn}(\theta - \vartheta))dt, \]

(9)

where \( \alpha \) and \( \beta \) are the proportional and integral gains of the dual-relay respectively [6], [7]. Figure 4 shows an overview of this voice-coil model under dual-relay feedback. Notice that (7) contains position dependent parameters. In order to improve the accuracy of identification, it is recommended to limit \( A < |\theta - \vartheta| \) to ensure the inequality below holds,

\[ |\cos \theta - \cos \vartheta| < \delta |\cos \vartheta|, \]

(10)

where \( 0 < \delta \ll 1 \). For the purposes of the experiments to be described later, \( \delta = 0.01 \) in order to limit the error bound of \( \hat{a}, \hat{b}, \hat{c} \) and \( \bar{d} \) to about 1% during oscillation excited by dual-relay. Figure 5 shows the allowable oscillation amplitude in different equilibrium positions from \(-60^\circ\) to \(60^\circ\). The maximum allowable amplitude is the lower bound of these two curves. It can be deduced from Figure 5, as \( \vartheta \) angle increases, the change of the force constant is more sensitive to a change in angle. Thus, the amplitude of oscillation \( A \) should be kept small and within the bounds specified by the error that the user and control system is able to tolerate. For instance, to ensure a 1% variation in \( \cos \theta \) at \( \vartheta = 0^\circ \), \( A < 8^\circ \) and at \( \vartheta = 30^\circ \), \( A < 1^\circ \). Even though the amplitude should be kept small, it is advisable that the amplitude of oscillation is adequate for data acquisition. Extremely small amplitude oscillations with a low frequency can possibly pose a tuning issue and could be challenging to hardware such as the resolution of the position encoder and data acquisition equipment.

The describing function of the nonlinear components, including the dual-relay and the Coulomb friction of the plant is given by,

\[ N(A) = \frac{4\alpha}{\pi A} - j\frac{A (\beta - \bar{c})}{\pi A}, \]

(11)

Using the harmonics balance condition \( N(A) = -1/G(j\omega) \), and excitation about the equilibrium point \( [\vartheta, 0]^T \) with frequency \( \omega \) and a sufficiently small amplitude \( A \) which satisfies the condition (10), yields,

\[ \hat{a} = \frac{1}{2} \left( \frac{4\alpha_1}{\pi A_1 \omega_1^2} + \frac{4\alpha_2}{\pi A_2 \omega_2^2} \right) = \hat{a}/\cos \vartheta, \]

(12)

\[ \hat{b} = \frac{4(\beta_2 - \beta_1)}{\pi (A_2 \omega_2 - A_1 \omega_1)} = \hat{b} \cos \vartheta, \]

(13)

\[ \hat{c} = \frac{\beta_1 A_2 \omega_2 - \beta_2 A_1 \omega_1}{A_2 \omega_2 - A_1 \omega_1} = \hat{c}/\cos \vartheta \]

(14)

where \( (\bullet) \) denotes the estimation of \( (\bullet) \) and \( (\bullet)_i \) is the value of \( (\bullet) \) in the \( i \)th experiment accordingly.

With dual-relay excitation of limit cycles at equilibrium point \( \theta = \vartheta \), the parameters of \( a, b \) and \( c \) can be efficiently identified by (12)–(14).

### III. Controller Design

As with a conventional motion control system, a composite controller with model-based “feedforward” and feedback are used for tracking of a trajectory \( (\dot{\theta}_d, \dot{\theta}_d, \ddot{\theta}_d) \), with identified \( \hat{a}, \hat{b}, \hat{c} \).

The combined control effort without angle compensation, \( \ddot{i} = i_m + i_d + i_s \), where the control effort \( i_m, i_d \) and \( i_s \) refers to the control effort of the model-based controller, the time-delay compensator and the sliding mode controller respectively. In order to cater to the position dependent non-linearity, the effective control effort, \( \ddot{i} = \ddot{i}/\cos \theta \).
The model-based controller is designed based on the parameters obtained through relay identification,
\[ i_m = \hat{a}\ddot{\theta} + \hat{b}\cos^2 \theta \dot{\theta} + \hat{c}\text{sgn}(\hat{\theta}_d), \] (15)
where the term \( \hat{b}\cos^2 \theta \) contains sensor feedback information. Set \( \hat{\theta} = \theta_d - \theta, \dot{\theta} = \theta_d - \theta, \) and \( \dot{\theta} = \hat{\theta} - \hat{\theta}. \) Substitute (15) into (6), we have
\[
\dot{\hat{a}} \hat{\theta} + \hat{b}\cos^2 \theta \hat{\theta} = (a - \hat{a})\dot{\theta}(t - \tau) + (b - \hat{b})\cos^2 \theta(t - \tau) \dot{\theta}(t - \tau)
+ c\text{sgn}(\theta(t - \tau)) - \hat{c}\text{sgn}(\hat{\theta}_d(t - \tau)) + d - i_d - i_s. \] (16)

Define \( \delta = (a - \hat{a})\dot{\theta}(t - \tau) + (b - \hat{b})\cos^2 \theta(t - \tau) \dot{\theta}(t - \tau)
+ c\text{sgn}(\theta(t - \tau)) - \hat{c}\text{sgn}(\hat{\theta}_d(t - \tau)) + d + i_d - i_s. \) Let \( i_d = \delta(t - \tau) \) to eliminate slow-varying disturbances.

With application of model-based controller and time-delay compensator, the error dynamics become
\[
\dot{\hat{a}} \hat{\theta} + \hat{b}\cos^2 \theta \hat{\theta} = \delta(t - \tau) - \delta(t - \tau) - i_s \] (18)

Define the sliding surface \( \sigma = \lambda\hat{\theta} + \hat{\dot{\theta}}. \) Consider a Lyapunov function candidate \( V = \lambda^2\hat{\theta}^2/2 \)
\[
\dot{V} = \dot{\hat{a}}\lambda\dot{\theta} + \hat{b}\cos^2 \theta \dot{\theta} \dot{\theta} + \hat{c}\text{sgn}(\theta) + k\lambda, \]
\[
\sigma = \dot{\hat{a}}\lambda\dot{\theta} + i_s = k\lambda \dot{\theta} + (k + \hat{a}\lambda - \hat{b}\cos^2 \theta) \dot{\theta} + \hat{c}\text{sgn}(\theta). \] (19)

If \( |\delta(t) - \delta(t - \tau)| < \varepsilon, \) then let
\[
i_s = (k + \lambda\dot{\theta} - \hat{b}\cos^2 \theta) \dot{\theta} + \varepsilon\text{sgn}(\sigma) + k\lambda, \]
\[
= k\lambda \dot{\theta} + (k + \hat{a}\lambda - \hat{b}\cos^2 \theta) \dot{\theta} + \hat{c}\text{sgn}(\theta). \] (20)

We have \( \dot{V} \leq -k\sigma^2. \) Thus, \( \sigma \rightarrow 0, \) then \( \dot{\theta} \rightarrow 0. \)

In summary, with estimation of \( \hat{a}, \hat{b}, \hat{c}, \) the following controller \( i(t) \) ensure tracking of arbitrary smooth reference signal \( (\theta_d, \dot{\theta}_d, \hat{\theta}_d). \)
\[
i(t) = \hat{i}(t)/\cos \theta(t), \]
\[
\hat{i}(t) = i_m(t) + i_d(t) + i_s(t), \]
\[
i_m(t) = \hat{a}\ddot{\theta} + \hat{b}\cos^2 \theta(t)\dot{\theta} + \hat{c}\text{sgn}(\hat{\theta}_d), \]
\[
i_d(t) = \hat{i}(t - \tau) - \hat{a}\ddot{\theta}(t - \tau) - \hat{b}\cos^2 \theta(t - \tau)\dot{\theta}(t - \tau)
- \hat{c}\text{sgn}(\hat{\theta}_d(t - \tau)), \]
\[
i_s(t) = k\lambda \dot{\theta} + (k + \hat{a}\lambda - \hat{b}\cos^2 \theta) \dot{\theta} + \hat{c}\text{sgn}(\theta). \] (26)

**Remark 3.1:** By using the controller (23)-(26), there are four parameters that require tuning, that is, \( k, \lambda \) and \( \varepsilon \) in \( i_s(t), \) and \( \tau \) in \( i_d(t). \)

**Remark 3.2:** In practice, in order to avoid excessive chattering, the \( \text{sgn}(\sigma) \) function in (26) is replaced with \( \text{sat}(\sigma/\eta), \) where \( \eta \) is a small positive number.

In order to estimate the two unknown states \( \hat{\theta} \) and \( \dot{\theta}, \) a high gain observer (HGO) with a sliding mode feature was proposed in [8]. The HGO was intended to keep the estimation error bounded while the sliding mode portion eliminated that bounded error. Our proposed controller is able to reject slow time-varying disturbances. Hence, it is able to accept a non-zero bounded estimation error. Therefore, the sliding mode portion of the observer has been omitted. The observer is given below,
\[
\dot{\hat{\Theta}} = A\hat{\Theta} + \psi(\hat{\Theta}) + \gamma(\hat{\Theta})u + L(y - C\hat{\Theta}) \] (27)
where \( \hat{\Theta}^T = [\hat{\theta}, \dot{\hat{\theta}}], \) is the state estimation vector of \( \theta, \) \( L \) is the HGO gain matrix and matrices \( A \) and \( C \) are given below,
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
In our case, the observer design is as follows,
\[
\psi(\hat{\Theta}) = \begin{bmatrix} 0 \\ \hat{\theta} - \hat{\theta} \cos^2 \hat{\theta} \end{bmatrix}, \quad \gamma(\hat{\Theta}) = \begin{bmatrix} 0 \\ \frac{1}{b}\cos(\hat{\theta}) \end{bmatrix} \]
where the constant \( \theta_s \) is the HGO gain.

**IV. Experiments**

**A. Set Up**

An integral linear-rotary actuator for high-speed pick-and-place applications has been developed by SIMTech [9]. The 2 pole, single phase Halbach array is incorporated in the rotary positioning module of the set-up [10]. The positioning module of 2-DOF linear-rotary actuator as shown in Figure 6 will be used for the experiments detailed in this section.

The rotary module is driven by a Copley Accelus driver with a 24V DC power input. A ±10V analog command signal is sent from the dSPACE DS1104 control card to the driver. This is used for identification of the plant model and for the implementation of the controller.

A 3rd order minimum jerk profile [11] is used for the purposes of this experiment which is shown in Figure 7.

**B. Relay Identification**

In this short section, the results of the relay identification procedure carried out on the actual plant will be presented.

The relay identification procedure is conducted at the equilibrium point of \( \dot{\theta} = 0 \) with a sampling rate of 100μs. Two sets of gains of relay and the corresponding limit cycle amplitudes and frequencies are listed in Table I and Figure 8. It should be noted that the both oscillations are within 8° or 0.14rad to ensure error on the nonlinear position dependent terms is kept to under 1%. There is a noticeable bias that exist in the relay oscillation which cannot be eliminated by the setting of the relay gains \( \alpha \) and \( \beta. \) This is most likely due to the inherently non-uniform, asymmetric.

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magnetic field distribution within. By (12)–(14), the model parameters are found and shown in Table II. The composite controller proposed will be compared against a traditional PID controller with a feedforward element. Both controllers will be manually tuned to minimize tracking and steady state errors.

The PID controller with a model-based feedforward element is given below,

\[ i = \bar{i}/\cos \theta, \quad \bar{i} = i_m + i_b, \quad i_m = \hat{\theta}_d + \hat{\theta} \cos^2 \theta \hat{\theta}_d + \hat{\theta} \text{sgn}(\hat{\theta}_d), \quad i_b = k_p \hat{\theta} + k_i \int_0^t \hat{\theta} dt + k_d \dot{\theta}. \]

A 2nd Order Low-pass Butterworth filter of a corner frequency of 200Hz is used to assist in stability of the system.

C. Composite Controller Parameter Settings

The composite and PID controllers follows the forms listed in equations (22)-(26) and (28)-(31) respectively. Both controller parameters are listed below in Table III. It should be noted that the measured value for position and estimates for velocity and acceleration were used as feedback. Tuning of the controller begins by setting \( \eta \) sufficiently small and \( \tau = 0 \), then \( k \) and was tuned for stability. Next, \( \lambda \) and \( \epsilon \) are tuned for the desired rate of error convergence. Lastly, adjustments can be made to \( \tau \) to increase rejection of slow time-varying disturbance, \( \eta \) to decrease chattering and possibly \( k \) to improve transient performance.

D. Comparison of Controller Performance

Data from both controllers were taken from the first cycle with zero initial conditions. Figure 9 shows a comparison between the performance of the composite and PID with feedforward. It also gives the notation of the stages of the profile. The control effort of both controllers are largely similar in magnitude especially in acceleration and deceleration. However, the tracking error of the composite controller is
much lower than that of the PID. Table IV shows in more definite terms that the composite controller can yield up to 9 to 13 times lower maximum tracking error. In the last segment (H to I), both controllers perform on par. This could be caused by non-uniformity of the magnetic field due to manufacturing imperfections and observer estimation errors.

It is noted that the performance of the PID controller could be impacted by the implementation of a 2nd order Butterworth filter. However, experiments could not be completed without the presence of the filter as the system is unstable otherwise. The integral term is noted to be not sufficient to eliminate steady state error. However, increasing the integral gain also increases tracking error. The observer’s position estimation performance in this application is good. The absolute peak and RMS position estimation error over a single cycle of the reference profile are 0.0065rad and 0.0024rad respectively. Since both velocity and acceleration were unmeasured, the performance of the observer on those two states cannot be determined.

V. CONCLUSION

The single-phase rotary servo motor with PM Halbach array design offers large driving torque. However, due to its nonlinear torque constant, back-EMF force and friction in the bearing, high-speed tracking control of the motor is challenging and there is no existing literature on the topic. In this paper, the dynamics of this type of servo motor is modeled with considerations of the nonlinearities above. A closed-loop dual-relay feedback unit is used to identify the model parameters through generation of a suitable steady oscillation. In the later part of the paper, a composite model-based controller with time-delay and sliding mode control elements is proposed to track the typical motion profiles despite parameter error and external disturbances.

Through experimental data and verified performance of the control, appropriate conclusions can be drawn that performance of this controller exceeds that of traditional PID with model-based feedforward element.

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