A DUOPOLISTIC MODEL OF

DYNAMIC COMPETITIVE ADVERTISING AND EMPIRICAL VALIDATION

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Abstract

This paper presents a differential game theory model for competitive advertising by extending the familiar Vidale-Wolfe single-decision-maker model of sales response to advertising. This model further extends the Deal (1979) model and the Lanchester model in the literature, which have been considered as competitive extensions of the Vidale-Wolfe model. The model is validated by using advertising and market share data for (a) Coke and Pepsi and (b) Marlboro and Winston. Model fitting and forecast accuracy are used in the validation study to compare its performance with other models. Necessary conditions for open-loop and closed-loop Nash equilibrium solutions to the model are derived. Because closed-form solutions are difficult to obtain, a numerical algorithm is developed. Characteristics of optimal (Nash Equilibrium) advertising strategies are discussed by using the example of Coke and Pepsi. Finally, some observations on the modelling of competitive advertising decisions are made.

INTRODUCTION

There has been a growing interest in the problem of determining profit-maximising advertising strategies by both practitioners and researchers in the areas of management science, marketing and economics. There are probably two reasons. First, it is a realistic problem faced by all firms using advertising to promote the sale of their products. Second, it is a very challenging problem because great complexity is involved in modelling. Accordingly, numerous models have been developed to study advertising decisions (Feichtinger et al 1994; Rao 1990; Feichtinger and Jorgensen 1983; Sethi 1977).

Advertising decisions are likely to be made in competitive situations where competitors attempt to increase their own market share at the expense of others. Thus, effective advertising strategies have to recognise rivals' competitive challenges. On the other hand, effective advertising strategies have to respond to changes in market conditions and evolve with time and, thus, are inevitably dynamic. As such, differential game theory models have been used to study competitive advertising decisions. A differential game theory model treats each competitor as a player in a competitive game and models the instantaneous variation of its market share as a function of the advertising expenditure levels and market positions of itself and its competitors. This has been generally accepted as a viable approach for the study of competitive advertising decisions.

A number of differential game theory models has been developed to study competitive advertising decisions in the last two decades. These models can be classified into two categories according to the type of product under investigation: durable products and non-durable products. Durable products are products that have a long usable life and each customer buys at most once in the planning interval. Non-durable products are products that have a relatively short usable life and each customer consumes, as normally assumed in advertising models, one unit in each time unit in the planning interval. For durable products, an advertiser's concern is only the potential customers who have not adopted the product. For non-durable products, an advertiser's concern is the potential customers who are not buying the product in the current period, either having or having not adopted it before. This is the fundamental difference between the two types of advertising decision models. A few differential game theory models have been developed to study competitive marketing decisions for durable products (Teng and Thompson 1983; Thompson and Teng 1984; Eliashberg and Chatterjee 1985).

The recent interest seems to have shifted from durable products to non-durable products. This is reasonable as every product can be considered as a non-durable product depending on the length of the planning interval. However, the modelling of competitive advertising decisions for non-durable products is still limited. The Lanchester model of combat appears to be the only model that has attracted much attention in this area. Although this model captures the dynamics of competition between two rivals in a simple way, it also seems to suffer from some drawbacks. First, it models a "steady state" market in which the total market share of the two competitors does not change. In reality, escalating advertising by competitors is likely to be seen in a growing market in which all competitors' market shares can increase (Erickson 1985). Second, advertising is modelled as the sole cause of market share variation, although market share changes are also caused by other factors such as product quality, service or promotional activities of other small competitors, to name a few.

In this paper, we propose a competitive advertising model for non-durable products. The model is based on the single-decision-maker advertising model developed by Vidale and Wolfe (1957) (referred to as the VW model thereafter). The VW model has a wide appeal because it was developed on the basis of empirical evidence and is simple enough for real applications. However, it lacks the competitive element of a real market. As a result, Deal (1979) extended it to a duopoly. The Lanchester model of combat, when used to study advertising decisions, has also been considered to be an extension of the VW model (Little 1979; Chintagunta and Vilcassim 1992). Our model can be considered as a further extension of the Deal model and the Lanchester model. The proposed model will be validated with empirical data by comparing its performance with other models. A duopoly is considered because (a) the interest in many competitive situations is focused on the two major rivals (Erickson 1985); and (b) much insight can be gained from the duopoly case and yet the model is relatively tractable from a mathematical standpoint.

The paper is organised as follows. In Section 2, our model is formulated by establishing the relationships of the impact of advertising on market share. An empirical validation exercise is included. In Section 3, open-loop and closed-loop Nash equilibrium advertising strategies are discussed and a numerical algorithm is developed. Conclusions and future research directions are addressed in Section 4.

MODEL FORMULATION

We first give a brief review of the VW model (1957) and its extensions to competitive settings. This will allow us to examine the justification of our model and the differences between our model and other models.

The VW Model and Its Extensions to Competitive Settings

The VW model was one of the earliest models to address the dynamics of advertising decisions. Based on empirical data, Vidale and Wolfe (1957) observed that changes in the rate of sales of a product depend on two effects: (a) response of advertising that acts on the unsold portion of the market; and (b) sales decay of the sold portion of the market due to factors such as product obsolescence, and competing advertising. The evolution of sales level in time *t* is modelled by

$$\frac{dS(t)}{dt} = a(1 - \frac{S(t)}{M}) - rS(t), \tag{1}$$

where $S(t)$ is the sales level at time t; M is the size of the potential market or saturation level; and r is a decay constant. The objective is to maximise cumulative sales profit after advertising expenditures.

It is assumed in the model that the effect of advertising is proportional to advertising expenditure or $a = kA(t)$, where $A(t)$ is the advertising expenditure at time t and k is the response rate to advertising. The model has been solved for optimal controls for advertising expenditure levels by Sethi (1973). Because later research has demonstrated diminishing returns to scale of advertising expenditure (Hanssens et al 1990), Sethi (1983) further modified the model to include this effect.

The VW model has a wide appeal because of its simplicity and empirical foundation. The drawback of the model is that competition is not directly reflected in the formulation, although the effect of competitive promotional activities can be indirectly reflected in the sales decay constant. Consequently, Deal (1979) extended it into a duopoly by using a differential game. The market share dynamics are modelled by

$$\frac{dx_i(t)}{dt} = k_i u_i(t)[1 - x_1(t) - x_2(t)] - r_i x_i(t), \; i = 1, 2 \qquad (2)$$

where subscript i is for player i; $x_i(t)$ is the market share of player i at time t; $u_i(t)$ is a variant of advertising expenditure for player i at time t; k_i is the response rate to advertising for player i; and r_i is the decay constant for player i. The objective was modified from the VW model to maximise cumulative sales profit plus a valuation of the ending market share for each player. To reflect the effect of diminishing returns to scale of advertising expenditure, advertising expenditure is modelled by $[u_i(t)]^2$ in the payoff function where $u_i(t) = \sqrt{A_i(t)}$ and $A_i(t)$ is the advertising expenditure at time t for player i.

The Deal (1979) model has received little attention in later research on competitive advertising. This is probably because it overlooks an important fact. It can be observed from (2) that advertising acts only on the unsold portion of the market, that is, each player's advertising does not directly affect the rival's market share. This is not consistent with empirical studies (Telser 1962; Little 1979; Carpenter et al 1988) that show competitive advertising can have a significant negative impact on a firm’s market share. In particular, from (2), the effect of advertising diminishes when $1-x_1(t)-x_2(t)$ approaches zero. Thus, no advertising should be used when the market is near saturation. This is unrealistic. Intense advertising competition often occurs in “saturated” markets and thus the model does not adequately capture the dynamics of competitive advertising.

The only model that has attracted more attention in the study of competitive advertising decisions for non-durable products is the Lanchester model of combat. The Lanchester model was originally developed to study the problem of combat. Kimball (1957) was among the first to note that a variation of the formulation could be used to model competitive advertising decisions. The formulation was further formalised by Little (1979). Subsequently, variations of this formulation have been used to study competitive advertising decisions (Case 1979; Erickson 1985, 1991, 1992; Sorger 1989; Chintagunta and Vilcassim 1992). Little (1979) claims that the Lanchester model can be considered as an extension of the VW model.

Under this model, market share shifts between the two rivals are modelled by

$$\frac{dx_i(t)}{dt} = k_i u_i(t)[1 - x_i(t)] - k_j u_j(t) x_i(t),\ i, j = 1, 2,\ i \neq j, \tag{3}$$

where $x_1(t) + x_2(t) \equiv 1$. The objective for each competitor is to maximise cumulative profit in the planning horizon.

In contrast to the Deal (1979) model, the Lanchester model captures the dynamics of competitive advertising by modelling advertising as the sole cause of market variation. It is assumed that the total market share of the two competitors does not change, or equivalently $x_1(t) + x_2(t) \equiv 1$, and advertising acts only on the market portion captured by the competitor. It can be seen from (3) that market shares do not change if both competitors do not advertise. This is the other extreme of the Deal (1979) model. The model does not appear to be appropriate for decision making purposes, particularly in growing markets where intense advertising competition is likely to occur. For example, it does not include the sales decay constant in the VW model that is used to capture the effects of product quality, service or competitive advertising by other small players.

The Model

We propose a model as an extension of the VW model. Market share evolution of the two competitors is modelled by

$$\frac{dx_i}{dt} = k_i u_i (1 - x_i) - (k_j u_j + r_i) x_i,\ x_i(0) = x_{i0},\ i, j = 1,2,\ i \neq j, \tag{4}$$

where the notations are the same as above, and the time argument for $x_i(t)$ and $u_i(t)$ is suppressed for clarity of expression.

The formulation (4) is different from the Deal model (2) and the Lanchester model (3). First, advertising acts on both the unsold market portion and the market portion captured by the competitor. This differentiates our model from the Deal model (2). Second, a restriction to a "steady state" market does not exist. Third, market share changes due to factors other than advertising (such as product quality, service and other small competitors) are captured by a sales decay constant. The second and third characteristics differentiate our model from the Lanchester model (3).

Due to its simplicity and adoption in other studies, the square root function is used to model the effect of advertising in our model. A different decay constant (r_i) is used for each player to reflect differences in product quality, services, etc. Similarly, a different advertising response rate (k_i) is used for each player to reflect differences in media selection and other marketing variables.

Each firm is to maximise its cumulative profit in the planning interval plus a valuation of its ending market share or

$$\max_{A_i \geq 0} J_i = \int_0^T (Mp_i x_i - u_i^2)dt + g_i x_i(T), \qquad (5)$$

where T is the length of the planning interval, that is, the planning interval is [0, T]; and p_i is the unit profit margin for player i.

A discount rate is not included in the objective function, although this does not add more complexity into our model. If a discount rate is included, the profit of the potential market (Mp_i) and advertising expenditure (u_i) should be expressed in the current value form. Note that the potential market profit (Mp_i) for each competitor is treated as a constant in the model. This condition is likely to hold in the present value case but not in the current value because of the effect of inflation on sales prices and costs of labour, material, etc. On the other hand, when planning is performed at the beginning of the planning horizon, the concern of the model is the true (present value) level of advertising expenditure. Expressing all values in the present value form makes solutions to the model easier to understand. Later discussions in this paper will be in present value without further elaboration.

Empirical Model Validation

Empirical data were used to validate the proposed model (4). The validation exercise focuses on the state equations of each model. This is because the state equations determine the fundamental relationship between market share changes and advertising expenditure. The objective function affects only the optimal solution of the model. We consider the fitting of the three models to the empirical data. If a model describes the true response relationship of market share to advertising expenditure, it should be able to closely explain or predict market share variations. Model fitting and forecast accuracy were used to evaluate the performance of the three models.

Two sets of advertising and market share data were used in the validation exercise: (a) Coke and Pepsi in the beverage market for the period 1969 to 1983; and (b) Marlboro and Winston in the cigarette market for the period 1983 to 1994. The data were collected from various issues of the *Advertising Age* magazine which reports annual sales, earnings, advertising expenditure and market shares for the 100 top advertisers in the United States every year. Market share is obtained by dividing the total annual (unit) sales of the product by the total annual (unit) sales of the market. Advertising expenditure each year is the amount of measured advertising dollars allocated to support each brand name. Each brand name includes all the products sold under the same brand name. For each data set, the two products are the two major brands that have been engaged in an intense advertising competition. The period for each data set is the period that reliable data could be collected by us.

Because market share and advertising data collected for the study are on an annual basis and no spending patterns of advertising within a year were available, the following discrete-time analog of the state equations is used to estimate the model parameters.

$$\begin{aligned} y_i(t+1) &= x_i(t+1) - x_i(t) \\ &= k_i u_i(t+1)[1 - x_i(t)] - [k_j u_j(t+1) + r_i]\, x_i(t), \quad i, j = 1,2, \; i \neq j. \end{aligned} \tag{6}$$

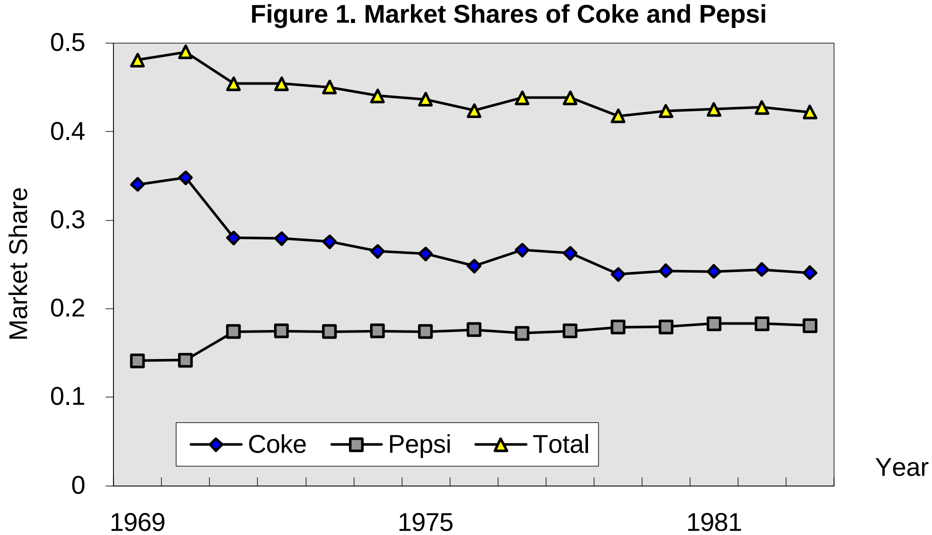
When advertising expenditure is constant through a year, an explicit solution can be obtained for the state equations of each model. We have verified that the numerical differences between the explicit solution and the discrete-time analog are negligible for all the three models. Therefore, the discrete-time analog is numerically equivalent to assuming that advertising within a year is evenly distributed. The discrete-time analog equation (6) can be easily modified for the Deal model and the Lanchester model.

For our model and the Deal model, a single estimation was used for the two equations of the two products because they share common parameters. For the Lanchester model, actual market shares of the two products were converted to relative market shares between the two products for the regression analysis.

Coke and Pepsi

Market share and advertising data for Coke and Pepsi have been used to compare open-loop and closed-loop Nash equilibrium solutions for the Lanchester model (Chintagunta and Vilcassim 1992, Erickson 1992). However, we use the data in this study to test model validity. The market shares of Coke and Pepsi in the beverage market are depicted in Figure 1.

Figure 1. Market Shares of Coke and Pepsi



Each model was fitted to different subsets of the data. Table 1 presents the results of regression for the last five years of the sample period. In each case, very similar advertising response rates and decay constants were obtained for Coke and Pepsi. This implies that customers responded to advertising of Coke and Pepsi in the same manner and the two products had compatible customer loyalty. This is reasonable given that the two products are very similar and the same media selections were normally used by the two companies.

The model significance is low for this data set. None of the models is significant in any case at the significance level of 5%. Although this may be a weakness of this empirical test, it is consistent with other empirical studies (Chintagunta and Vilcassim 1992). For individual model performance, the results are very consistent. In every case, the Lanchester model has the highest value of R^2 and the lowest significance probability (p), followed by our model, and then the Deal model.

Table 1. Empirical Model Estimation -- Coke and Pepsi

Data	Parameter Product		Deal Model Parameter (Sig.)	Deal Model R^2/p	Lanchester Model Parameter (Sig.)	Lanchester Model R^2/p	Our Model Parameter (Sig.)	Our Model R^2/p
1969-1983	Coke	k_1	.01451 (.174)	.243	.02563 (.118)	.344	.00984 (.192)	.283
		r_1	.14689 (.095)				.09677 (.253)	
	Pepsi	k_2	.01146 (.325)	.140	.02145 (.055)	.080	.00960 (.128)	.081
		r_2	.12531 (.382)				.12007 (.317)	
1969-1982	Coke	k_1	.01459 (.204)	.242	.02842 (.131)	.352	.01012 (.211)	.286
		r_1	.14739 (.116)				.09512 (.291)	
	Pepsi	k_2	.01223 (.367)	.174	.02300 (.066)	.092	.01062 (.142)	.102
		r_2	.13574 (.425)				.13996 (.314)	
1969-1981	Coke	k_1	.01375 (.280)	.244	.03153 (.156)	.359	.00990 (.269)	.292
		r_1	.14164 (.166)				.08834 (.365)	
	Pepsi	k_2	.01341 (.166)	.210	.02472 (.084)	.108	.01188 (.150)	.124
		r_2	.15202 (.446)				.16769 (.296)	
1969-1980	Coke	k_1	.01317 (.341)	.254	.03835 (.147)	.382	.00998 (.302)	.313
		r_1	.13786 (.209)				.07553 (.468)	
	Pepsi	k_2	.01789 (.347)	.236	.02841 (.085)	.115	.01527 (.123)	.131
		r_2	.21402 (.389)				.23929 (.220)	
1969-	Coke	k_1	.01079 (.490)	.275	.03680 (.186)	.398	.00775 (.473)	.334
		r_1	.12166 (.317)				.05048 (.661)	

1979	Pepsi	k_2 r_2	.02348 (.291) .27952 (.326)	.244	.02804 (.106)	.131	.01667 (.118) .27060 (.197)	.241

One difficulty encountered in using the model fitting results in Table 1 to compare individual model performance is that a direct comparison, particularly of the value of R^2, between the Lanchester model and the other two models may generally not be meaningful. Because relative market shares were used in the regression analysis for the Lanchester model, market share variations explained by the Lanchester model are changes of the relative market positions of the two products. However, market share variations explained by the other two models are market share changes in the actual market. To compare the Lanchester model with the other two models, model forecast capability was considered.

Each model was used to forecast the market shares for the last seven years of the sample period. The discrete-time analog and parameters estimated with prior data were used. For example, market share changes in 1983 for our model were obtained by using (6) and the advertising expenditures of the two products in 1983. The model parameters were estimated by using regression analysis of data for 1969-1982 (Table 1). Actual market shares were obtained from the Lanchester model by multiplying the predicted relative market shares by the total actual market share of the two products in the previous year. The predicted market shares by each model were then compared with the actual observed market shares. Table 2 summarizes the mean squared errors (MSE) and the mean absolute deviation (MAD) of forecast.

Table 2: Forecast Accuracy - Coke and Pepsi

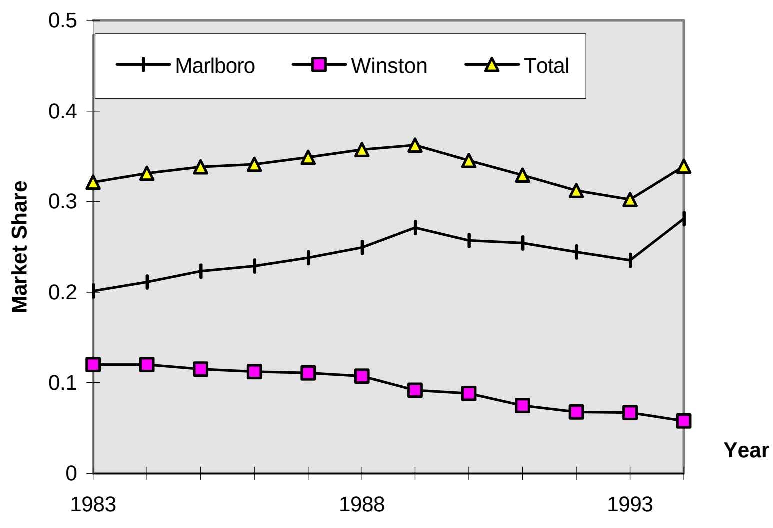
		Deal Model	Lanchester Model	Our Model
Coke	MSE MAD	.000221 .010282	.000185 .010413	.000211 .009711
Pepsi	MSE MAD	.000054 .006351	.0000297 .004698	.0000598 .006615
Overall	MSE MAD	.000137 .009465	.000107 .007995	.000135 .008814

According to forecast accuracy, all the three models gave very close predictions (less than 1% in MAD) to the observed market shares. The Lanchester model had the lowest forecast errors, followed by our model and then the Deal model. This is consistent with the model fitting results. However, the differences in MSE (< 0.00003) and MAD (< 0.0015) between the three models are practically negligible. The performance of the three models in forecast was relatively close.

Marlboro and Winston

Marlboro and Winston represent a slightly different situation from Coke and Pepsi. Their market shares in the cigarette market in the sample period are depicted in Figure 2. In the case of Coke and Pepsi, the two products were relatively symmetrical in terms of market share and advertising expenditure. Their total market share in the sample period was very stable except for a drop in 1971 (Figure 1). This is particularly suitable for the Lanchester model because of its assumption of a "steady state" market. However, Marlboro and Winston were less symmetrical, with Marlboro having a much higher market share and receiving a much higher advertising budget. Their total market share also exhibited more variations than that of Coke and Pepsi.

Figure 2. Market Shares of Marlboro and Winston



The models were also fitted into five different subsets of data. The results of regression are displayed in Table 3. The model effect is significant in every case at the significance level of 5%. Thus, each model was able to explain a significant portion of the market share variations. However, the model fitting results also reflect the non-symmetrical positions of the two products. The estimated model parameters for the two products are usually very different, with those for Marlboro being generally larger and more significant. In several cases, negative but highly insignificant (significance probability greater than 0.5) model parameter(s) for Winston was obtained for the full model. In such cases, one of the independent variables with negative coefficients for the full model was dropped from the regression and the best model (all parameters non-negative and the highest adjusted R^2) was used.

Table 3. Empirical Model Estimation -- Marlboro and Winston

Data	Product		Deal Model		Lanchester Model		Our Model	
			Parameter (Sig.)	R^2/p	Parameter (Sig.)	R^2/p	Parameter (Sig.)	R^2/p
1983-1994	Marlboro	k_1	.015517 (.0040)	.4863	.012122 (.3776)	.5474	.011248 (.0118)	.4415
		r_1	.354162 (.0072)				.274421 (.0227)	
	Winston	k_2	*[1]	.0047	.003854 (.6994)	.0282	.003095 (.5479)	.0263
		r_2	.0533 (.1095)				.095282 (.6812)	
1983-1993	Marlboro	k_1	.012622 (.0008)	.5836	.008879 (.4175)	.6026	.010006 (.0004)	.6281
		r_1	.297164 (.0012)				.266163 (.0005)	
	Winston	k_2	*	.0016	.002158 (.7853)	.0249	.000828 (.2006)	.0006
		r_2	.04983 (.00293)				*	
1983-1992	Marlboro	k_1	.011872 (.0031)	.5824	.006346 (.005)	.6472	.009539 (.002)	.6167
		r_1	.276209 (.0051)				.251908 (.003)	
	Winston	k_2	*	.0037	*	.005	.000712 (.3155)	.002
		r_2	.051451 (.0355)				*	
1983-1991	Marlboro	k_1	.01066 (.0125)	.5854	.006398 (.0097)	.6392	.008707 (.0094)	.6097
		r_1	.242106 (.022)				.22541 (.0158)	
	Winston	k_2	*	.008	*	.0097	.00061 (.4282)	.0055

[1]: The variable was dropped in the regression because of negative but highly insignificant parameter when the full model is applied.

		r_2	.04896 (.058)				*	
1983-	Marlboro	k_1	.009966 (.0244)	.6464	.037315 (.1309)	.733	.009562 (.0117)	.6739
		r_1	.221039 (.0447)				.217858 (.0305)	
1990	Winston	k_2	.006612 (.3539)	.0234	.024917 (.1881)	.0368	.006261 (.2992)	.0161
		r_2	.259125 (.2794)				.232607 (.3609)	

Between our model and the Deal model, ours had a higher value of R^2 and lower significance probability (p) in every case except the first one. This is consistent with the empirical results for Coke and Pepsi. The Lanchester model had the lowest model significance (or highest significance probability p) in every case, although it had the highest value of R^2 in four of the five cases.

For model forecast accuracy, market shares for Marlboro and Winston were obtained by each model for the last five years of the sample period, using the same method described for Coke and Pepsi. Table 4 summarises the mean squared errors and the mean absolute deviations. It can be observed that the Deal model had the lowest forecast errors. Our model gave marginally higher forecast errors than the Deal model. The Lanchester model had the highest forecast errors (MSE and MAD).

Table 4: Forecast Accuracy - Marlboro and Winston

		Deal Model	Lanchester Model	Our Model
Marlboro	MSE	.000439	.000534	.00048
	MAD	.016629	.019672	.015791
Winston	MSE	.0000215	.000133	.0000932
	MAD	.004062	.007604	.006889
Overall	MSE	.000230	.000333	.000287
	MAD	.010346	.013638	.01134

To summarise the findings of the empirical study for the two pairs of products, our model appears to be better than the Deal model. This is observed from our model's consistently higher value of R^2 and model significance than that of the Deal model for both data sets. According to forecast accuracy, each model performed marginally better than the other in one case.

The comparison of the Lanchester model and the other two models was difficult because relative market shares were used in model fitting for the Lanchester model. However, based on model significance and forecast accuracy, the Lanchester model appeared to be better for Coke and Pepsi but worse for Marlboro and Winston than our model and the Deal model. Given the differences of the two pairs of products in their respective market, it appeared that the Lanchester model may perform better if the "steady state" condition is well satisfied but worse if variations in the total market share of the two products are relatively large. Market share data reported in *Advertising Age* show that (a) the two major brands in a market usually have only a small share of the total market; and (b) "dominating" brands change after a certain period of time. In this sense, a "steady state" market is rare and the use of the Lanchester model is limited. Our model appears to perform better under more general market conditions (for example, the total market share varies).

SOLUTION TO THE MODEL

It is assumed that two competitors do not collaborate in advertising campaigns. Thus, a Nash equilibrium that characterises strategies in which no player can improve his position by unilaterally deviating from it is a proper solution. Nash equilibrium strategies have normally been used for other competitive advertising models.

The Hamiltonians for our model are given by

$$\begin{aligned} H_i &= M p_i x_i - u_i^2 + \lambda_i [k_i u_i (1 - x_i) - (k_j u_j + r_i) x_i] \\ &\quad + \phi_i [k_j u_j (1 - x_j) - (k_i u_i + r_j) x_j], \quad i, j = 1,2, \; i \neq j, \end{aligned} \tag{7}$$

where λ_i and ϕ_i $(i = 1,2)$ are auxiliary variables (Basar and Olsder 1982).

The necessary conditions for Nash optimality (open-loop or closed-loop strategies) are given by (Basar and Olsder 1982) the state equations (4) and

$$\frac{d\lambda_i}{dt} = -\frac{\partial H_i}{\partial x_i} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial x_i}, \ \lambda_i(T) = g_i, \ i, j = 1,2, \ i \neq j, \tag{8}$$

$$\frac{d\phi_i}{dt} = -\frac{\partial H_i}{\partial x_j} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial x_j}, \ \phi_i(T) = 0, \ i, j = 1,2, \ i \neq j, \tag{9}$$

$$u_i = \underset{u_i}{\arg \ Max} \ H_i. \tag{10}$$

From (10), we obtain

$$u_i = \frac{k_i[\lambda_i(1-x_i) - \phi_i x_j]\eta_i}{2}, \quad i, j = 1,2, \ i \neq j, \tag{11}$$

where η_i ($i = 1, 2$) is defined by

$$\eta_i = \begin{cases} 0 \text{ if } \lambda_i(1-x_i) - \phi_i x_j < 0, \\ 1 \text{ if } \lambda_i(1-x_i) - \phi_i x_j \geq 0. \end{cases} \tag{12}$$

Open-Loop Nash Equilibrium (OLNE) Strategies

Let us consider OLNE strategies first. Because u_i is a function of time alone, $\frac{\partial u_i}{\partial x_j} = 0$ for $i, j = 1, 2$. The conditions for the auxiliary variables (8) and (9) become

$$\frac{d\lambda_i}{dt} = -Mp_i + \lambda_i(k_1u_1 + k_2u_2 + r_i), \ \lambda_i(T) = g_i, \ i = 1, 2, \tag{13}$$

$$\frac{d\phi_i}{dt} = \phi_i(k_1u_1 + k_2u_2 + r_i), \ \phi_i(T) = 0, \ i = 1, 2. \tag{14}$$

It can be observed from (14) that if $\phi_i(t) > 0$ (< 0) for any $0 \leq t < T$, $d\phi_i(t)/dt \geq 0$ (≤ 0) and, hence, $\phi_i(T) > 0$ (< 0). This is contradictory to the terminal condition $\phi_i(T) = 0$. The only solution to (14) is then $\phi_i(t) = 0$. Similarly, we observe from (13) that $\lambda_i(t) > 0$ for $0 \leq t < T$. Hence, from (11) and (12), we obtain

$$u_i = \frac{k_i \lambda_i (1 - x_i)}{2}. \quad (15)$$

Therefore, OLNE advertising strategies are determined by equation (15), the two system equations (4) with two boundary conditions at the initial time ($x_i(0) = x_{i0}$, i = 1,2), and the two differential equations (13) for the auxiliary variables with two boundary conditions at the terminal time ($\lambda_i(T) = g_i$, $i = 1, 2$). For OLNE strategies, a firm's advertising is never zero.

Closed-Loop Nash Equilibrium (CLNE) Strategies

Let us now consider CLNE strategies. In this case, the control for advertising is a function of time and market share. Using (11), the conditions for the auxiliary variables (8) and (9) become

$$\frac{d\lambda_i}{dt} = -M p_i + \lambda_i (k_1 u_1 + k_2 u_2 + r_i) - \frac{1}{2} \kappa_j^2 \phi_j [\lambda_i x_i - \phi_i (1 - x_j)] \eta_j \quad (16)$$

with terminal conditions $\lambda_i(T) = g_i$, $i, j = 1, 2$, $i \neq j$, and

$$\frac{d\phi_i}{dt} = \phi_i (k_1 u_1 + k_2 u_2 + r_j) - \frac{1}{2} \kappa_j^2 \lambda_j [\lambda_i x_i - \phi_i (1 - x_j)] \eta_j \quad (17)$$

with terminal conditions $\phi_i(T) = 0$, $i, j = 1, 2$, $i \neq j$.

Therefore, CLNE advertising strategies are determined by equation (11); the two system equations (4) with two boundary conditions at the initial time ($x_i(0) = x_{i0}$, $i = 1, 2$), and the four differential equations (16) and (17) with four boundary conditions at the terminal time ($\lambda_i(T) = g_i$ and $\phi_i(T) = 0$, $i = 1, 2$). Under normal conditions, a firm's advertising expenditure will not be zero. However, this condition does not generally hold for CLNE strategies.

A Numerical Algorithm

After specifying the necessary and sufficient conditions for OLNE and CLNE strategies, optimal advertising trajectories can be obtained by solving a set of differential equations. For either OLNE or CLNE strategies, a closed-form solution is very difficult to obtain. A numerical algorithm is then developed by utilising the classical fourth-order Runge-Kutta method.

The algorithm starts with estimating an initial value for each auxiliary variable $\lambda_i(0)$ and $\phi_i(0)$ (for CLNE only). Then the differential equations for the necessary and sufficient conditions of Nash optimality are solved forward in time by using the fourth-order Runge-Kutta method (Tierney 1985). The values of the auxiliary variables at the terminal time, $\lambda_i(T)$ and $\phi_i(T)$, are compared with the target values g_i and 0, respectively. If they are within predetermined ranges, a numerical solution is obtained and the procedure is stopped. Otherwise, the differential equations are solved again with the initial values $\lambda_i(0)$ adjusted by δ_i given by

$$\delta_i = \alpha[g_i - \lambda_i(T)]e^{-\int_0^T U_i(x)dx}, \ 0 < \alpha < 1, \tag{18}$$

and the initial values $\phi_i(0)$ (for CLNE only) adjusted by σ_i given by

$$\sigma_i = -\alpha\phi_i(T)e^{-\int_0^T U_j(x)dx}, \ 0 < \alpha < 1, \tag{19}$$

where α is a parameter used to scale the adjustment term.

To obtain the adjustment terms in (18) and (19), let us consider equation (13) for OLNE strategies. By solving the differential equation (13), we obtain

$$\lambda_i(t) = [-Mp_i\int_0^t e^{-\int_0^y U_i(x)dx}dy + \lambda_i(0)]e^{\int_0^t U_i(x)dx}, \tag{20}$$

where $U_i(x) = k_1u_1+k_2u_2+r_i$, i = 1, 2.

The terminal values $\lambda_i(T)$ are given by

$$\lambda_i(T) = -Mp_i\int_0^T e^{\int_y^T U_i(x)dx}dy + \lambda_i(0)e^{\int_0^T U_i(x)dx}, \ i = 1, 2. \tag{21}$$

This result implies that, for OLNE strategies, the terminal value $\lambda_i(T)$ increases by $e^{\int_0^T U_i(x)dx}$ when the initial value $\lambda_i(0)$ increases by 1. Therefore, if the current advertising controls generate terminal values for the auxiliary variables $\lambda_i(T)$ different from the target values g_i, the initial value $\lambda_i(0)$ should be adjusted by $[g_i-\lambda_i(T)]\, e^{-\int_0^T U_i(x)dx}$.

For CLNE strategies, the solution (20) for $\lambda_i(t)$ does not hold. However, as we will discuss later on, the value of $\phi_i(t)$ is normally small compared with $\lambda_i(t)$. Thus, the auxiliary variables $\lambda_i(t)$ should change in a similar manner for OLNE and CLNE strategies. The same adjustment δ_i of $\lambda_i(0)$ for OLNE can be used for CLNE. For the auxiliary variable $\phi_i(t)$, its magnitude is usually small. From (17), a solution for $\phi_i(t)$ similar to the solution for $\lambda_i(t)$ given by (20) can be obtained, given all the other variables. A similar adjustment term to δ_i, with a slightly different integral for the exponent, can then be developed. For the simplicity of the algorithm, we use the adjustment term σ_i for $\phi_i(0)$ after some modification. This modification does not significantly affect the performance of the algorithm.

The performance of the algorithm depends on the initial values of the auxiliary variables. Appropriate estimates should be used for the convergence of the algorithm. For OLNE strategies, from (20), we obtain

$$\lambda_i(0) = Mp_i \int_0^T e^{-\int_0^y U_i(x)dx} dy + g_i e^{-\int_0^T U_i(x)dx}, \; i = 1, 2. \qquad (22)$$

An estimate of $\lambda_i(0)$ was obtained in our computation by using the actual average advertising expenditures. For CLNE strategies, the values of $\lambda_i(0)$ for the OLNE strategies (OLNE strategies were solved first) were used. An estimate for $\phi_i(0)$ was obtained by using (11) and the estimated values of $\lambda_i(0)$ and $u_i(0)$.

For every pair of optimal controls, it was checked against the Nash equilibrium property. For each player in turn, the value of its advertising control was perturbed while the competitor's advertising was fixed at the Nash equilibrium trajectory obtained from the numerical algorithm. The player's objective function value should not be improved and the competitor's objective function value should not be worse.

The algorithm exhibited stability and consistent convergence during extensive testing. Nash equilibrium solutions were consistently obtained. The algorithm converged fast (within a few iterations) for both OLNE and CLNE policies. In order to assess the performance of our algorithm, we also attempted to use the algorithm developed by Deal (1979). With the actual advertising expenditures as initial values of the controls, the algorithm did not exhibit good convergence properties in our trials.

Discussion of Results

We used the numerical algorithm to obtain optimal advertising strategies for Coke and Pepsi. The parameters estimated in the empirical validation exercise using the data of 1969-1983 were used in the study. To obtain numerical solutions to the problem, we further estimated that the potential market profit Mp_i for Coke and Pepsi was between $200 million and $300 million based on earnings of the two companies in the sample period. For simplicity, we considered cases where the unit profit margin was the same for the two products, or $p_1 = p_2$, and where the two competitors valued ending market share the same, or $g_1 = g_2$. Numerical solutions were obtained for two different market profits ($i = 1,2$): (a) Mp_i = $200 million, and (b) Mp_i = $300 million; and three different values for the ending market share ($i = 1,2$): (a) $g_i = 0$, (b) $g_i = 600$, and $g_i = 1200$. A value of 0.5 for α was used and an accuracy of less than 0.1 for the terminal values of all auxiliary variables was attained.

The OLNE and CLNE advertising policies of Coke and Pepsi for various cases are displayed in Figure 3.

Figure 3. OPNE and CLNE Policies

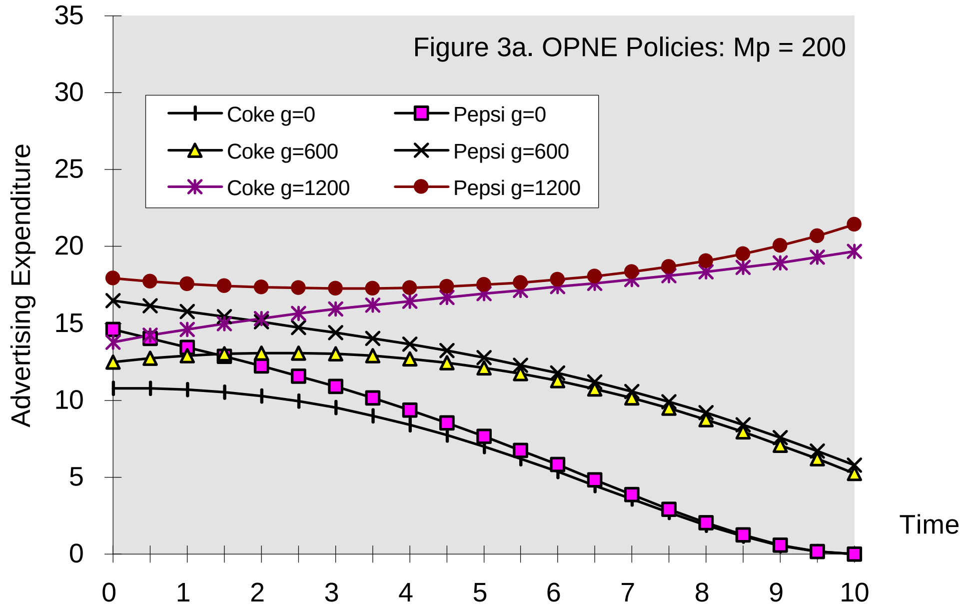


Figure 3a. OPNE Policies: Mp = 200
Coke g=0
Pepsi g=0
Coke g=600
Pepsi g=600
Coke g=1200
Pepsi g=1200
Advertising Expenditure
Time
0
5
10
15
20
25
30
35
0
1
2
3
4
5
6
7
8
9
10

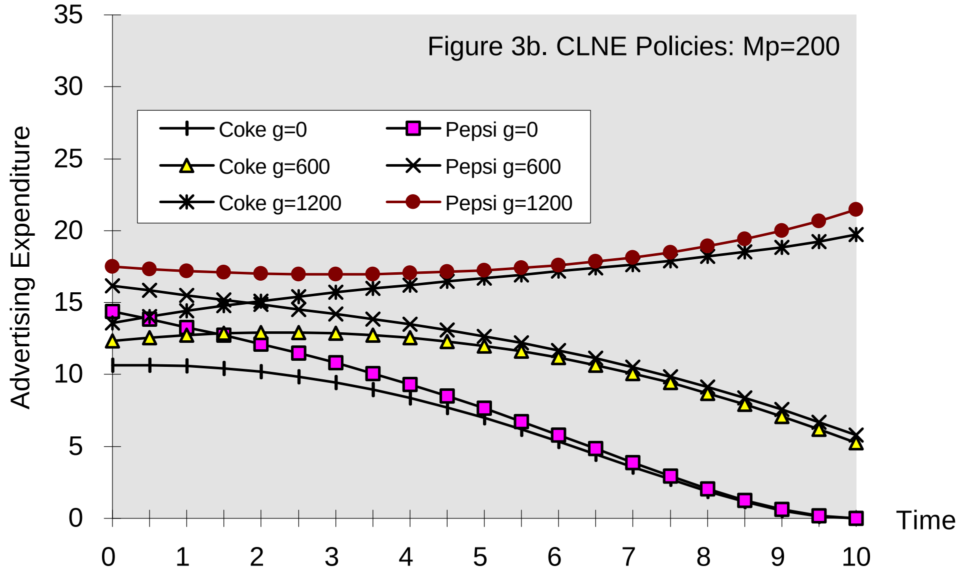
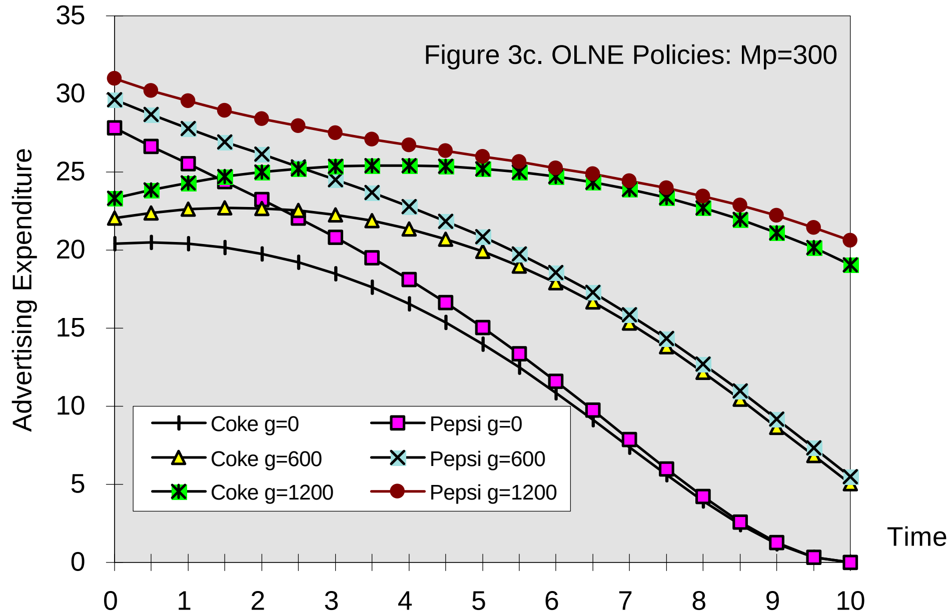
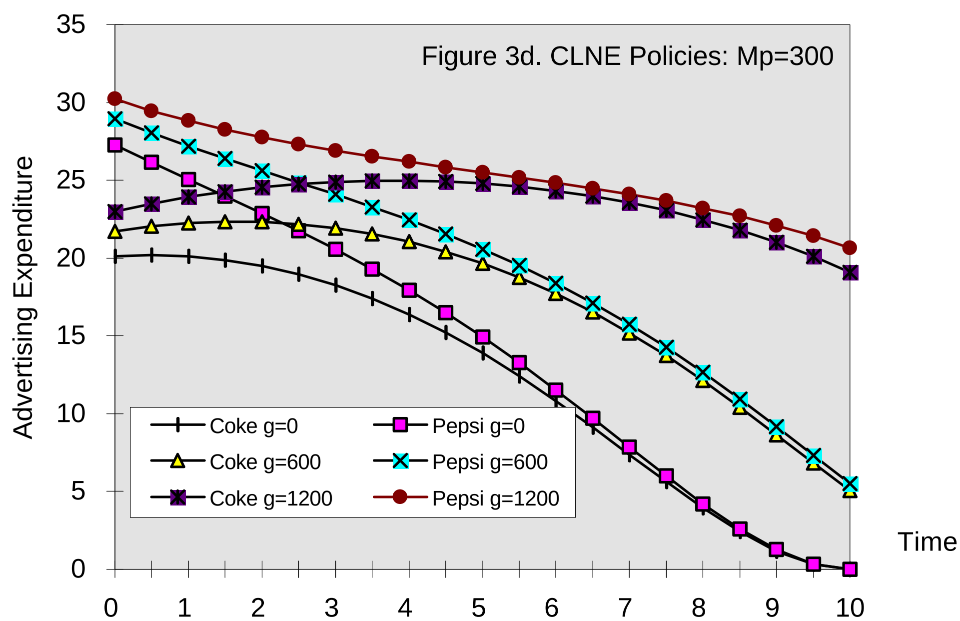


Figure 3b. CLNE Policies: Mp=200
Coke g=0
Pepsi g=0
Coke g=600
Pepsi g=600
Coke g=1200
Pepsi g=1200
Advertising Expenditure
Time
0
5
10
15
20
25
30
35
0
1
2
3
4
5
6
7
8
9
10





It is generally argued that CLNE strategies should be used for the study of competitive advertising decisions. In theory, CLNE strategies are superior to OLNE strategies. The two types of Nash equilibrium strategies are normally different (Jorgensen 1982). Our numerical results are consistent with these findings.

CLNE strategies are different from OLNE strategies and always yield a higher payoff than OLNE strategies for both competitors.

However, the difference in the objective function values is very small (less than 1%) in all the cases in our numerical study. It can be seen from Figure 3 that the OLNE and CLNE strategies are very similar in each case. This is different from the findings of Chintagunta and Vilcassim (1992) that CLNE strategies are very different from OLNE strategies. This is probably because (a) the two models are very different; and (b) the equilibrium solution used in their study is different from the long-term equilibrium determined at the beginning of the planning interval used in our study.

When market share is modelled as a result of foregone advertising and advertising policies determined at the beginning of the planning horizon, it is reasonable to expect OLNE and CLNE policies to be similar. The two types of polices differ in functional form specification in obtaining a solution. However, if they are determined at the beginning of the planning horizon, this difference does not produce major differences in the resulting advertising policies. To see this point, consider the conditions (11) for the two types of policies. The auxiliary variables $\phi_i(t)$ ($i = 1,2$) are always zero for OLNE policies but normally non-zero for CLNE policies. Economically, $\phi_i(t)$ is the change in competitor *i*'s profit from time *t* to *T*, with respect to a unit change in the rival's market share. At any time, this change is small for our model. On the other hand, $\lambda_i(t)$, or the change in competitor *i*'s profit from time *t* to *T* with respect to a unit change in its own market share, is usually very large compared with $\phi_i(t)$. The difference in u_i between the two cases will then be small. Thus, the two policies are similar.

For the two types of policies, the level of expenditure is mainly determined by the potential market profit level (Mp_i). It can be observed from Figure 3 that advertising expenditure increases significantly when the market profit increases from $200 million to $300 million. In contrast, the level of advertising expenditure increases only slightly when the value of the ending market share increases from $0 to $1,200 million.

However, the value of the ending market share determines the distribution of the advertising expenditure in the planning horizon. When the ending market share is not considered to be important (for example, when $g_i = 0$), advertising should be concentrated at the early stage of the planning interval. Nash equilibrium advertising trajectories are generally decreasing over time. This is sensible because market share gained at time t will generate profit from time t to T. If the ending market share is not valuable, the advertising induced market share increase should be obtained sooner rather than later. On the other hand, when the ending market share is considered to be important, time preference of advertising spending disappears because the effect of late advertising in the planning interval is captured by the ending market share. In this case, advertising expenditure should be spread more equally over the whole planning interval. In that way, the value of ending market share for each player should not be omitted from the formulation of dynamic competitive advertising models.

The market shares for the optimal advertising trajectories displayed in Figure 3 show that the market share of Coke generally decreases while the market share of Pepsi generally increases. This is because this trend was shown in the market share data used in the estimation of the model parameters and captured in the estimated model parameters.

CONCLUSION

We developed a model for the study of competitive advertising decisions in this paper. The model was based on the single-decision-maker model developed by Vidale and Wolfe (1957). The VW model has a wide appeal because of its simplicity and empirical origin, but lacks the competitive element of a real market. As a result, Deal (1979) extended it to a duopoly. The Lanchester model of combat, when used to study competitive advertising decisions, has also been considered to be an extension of the VW model. The model presented in this paper has fundamental differences from, and can be considered as a further extension of, the Deal model and the Lanchester model.

An empirical validation analysis was performed using two sets of empirical advertising and market share data: (a) Coke and Pepsi in the period 1969-1983; and (b) Marlboro and Winston in the period 1983-1994. The empirical study suggests that our model generally out-performs the Deal model. This is consistent with previous empirical studies which have shown that competitive advertising affects rival's market share. The Lanchester model appeared to perform better in the case of Coke and Pepsi where the total market share of the two products remained very stable; but worse than our model and the Deal model in the case of Marlboro and Winston where the total market share of the two products exhibited more variations. Given the different assumptions for the development of each model, the Lanchester model may perform well when the "steady state" market condition is satisfied but our model may perform better under more general market conditions.

Both OLNE and CLNE strategies for our model were discussed and the necessary and sufficient conditions derived. As closed-form solutions are very difficult to obtain, a numerical algorithm utilising the fourth-order Runge-Kutta method has been developed. Optimal advertising trajectories have been developed for Coke and Pepsi using empirically estimated model parameters. The numerically obtained optimal advertising policies for Coke and Pepsi shed some intuitively sensible insights into the nature of optimal advertising policies.

First, CLNE strategies always yield higher payoffs than OLNE strategies for both competitors. In this sense, CLNE strategies should be used for the study of competitive advertising decisions. However, the differences in payoff and advertising expenditure are very small. This is considered to be reasonable for our model if planning is done at the beginning of the planning horizon.

Second, for both OLNE and CLNE policies, the advertising expenditure level is mainly affected by the size of the potential market profit. A high level of potential market profit justifies a high level of advertising spending.

Finally, how a firm values its market share at the end of the planning interval is important for determining effective advertising strategies. Specifically, the value of the ending market share determines the advertising spending pattern in the planning horizon. When the ending market share is not considered to be important, advertising spending should be concentrated at the early stage. The optimal advertising trajectories are generally decreasing over time. When the ending market share is considered to be important, advertising spending should be more equally spread across the planning interval. In this sense, valuations of ending market shares should not be omitted from dynamic competitive advertising models.

FURTHER RESEARCH

Several important issues for future research of competitive advertising decisions for non-durable products arise from our study. First, the validity of competitive advertising models has not been established through empirical evidence. Although the current study provides some evidence on this issue, only two conveniently selected empirical data sets were used. More empirical studies are needed. Second, long-term solutions have normally been used in studies of competitive advertising decisions. However, firms simply do not commit to long-term strategies, whether they are open-loop or closed-loop. Therefore, strategies that can be used by competitors for short-term planning should be developed. Third, data reported in the *Advertising Age* show that there usually more than two "major" players in a market. In this sense, oligopolistic rather than duopolistic models should be developed.

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