Optimal cycle time and production rate in a family production context with shelf life considerations

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Abstract

This paper considers the problem of determining the optimal production rate for each item and the optimal cycle time for the family of items in a family production context with restrictions on the shelf life of the various items in the family. An algorithm that determines the optimal production rate for all the items as well as the optimal cycle time is proposed. This is an improvement over the existing method by Silver (1995) that can determine the optimal production rate only for one item for which the shelf life constraint is binding. Examples are provided to illustrate the proposed method.

1. Introduction

In this paper, we consider the problem of determining the production schedule of a family of items that share a piece of production equipment. All items in the family share the same cycle time T and are produced once every T units of time. This approach known as the family-production approach or the common-cycle time approach is popular in practice because of the ease of determining feasible production schedules.

Each items faces demand at a constant rate. The items are produced on a shared piece of equipment at a finite rate. The objective is to determine the production rates and cycle time, so as to minimise the total cost comprised of setup cost, holding cost and machine operating cost. The problem has been earlier addressed by Silver (1989, 1990, 1995), Sarker and Babu (1993), Goyal (1994) and Viswanathan (1995).

Silver (1990) considered a model without any shelf life constraint, in which there is the possibility of reducing the production rate of the items. Silver (1989) considers the problem where there is a shelf life constraint on the items that is violated by exactly one item. He considers the option of either slowing down the production rate for that item or reducing the cycle time to meet the shelf life constraint and proves that it is more effective to reduce the production rate of the item. Sarker and Babu (1993) consider the same model, and show that when the machine operating cost is incorporated into the model, it may sometimes be more effective to reduce the cycle time. Goyal (1994) pointed out that it may be cost effective to schedule the production of some of the items more than once in a cycle. However as pointed out by Viswanathan (1995), this might lead to infeasible schedules.

Silver (1995) considered a model with shelf-life constraint in which the optimal cycle time and the production rate for the item with a binding shelf life constraint are simultaneously determined. That is, instead of considering the option of reducing the cycle time or reducing the production rate of the particular item with shelf life restrictions, these are simultaneously optimised.

One weakness of Silver's (1995) model is that it can deal with the violation of the shelf life constraint for only one item. More over the production rate is optimised only for this item. It might be more cost effective to optimise the production rates of other items even if their shelf life constraint is not bindi ng. In this paper, we develop a method that removes these weaknesses. We propose an algorithm that obtains the optimum cycle time as well the optimum production rates for **all the items** for a situation in which more than one item might have a binding shelf life constraint.

2. Notations and model formulation

The assumptions made in this paper are the same as the assumptions made in Silver (1995), except for assumption (6). We assume that there can be more than one item with a binding shelf life constraint. For the sake of completeness, the assumptions are stated below:

(1) the demand rate for each item is known and constant;

(2) the stock is used on a FIFO basis;

(3) there is a known setup time for each item;

(4) every item is produced on each cycle;

(5) no deliberate shortages are allowed;

(6) there is a shelf life constraint on some of the items, and this constraint could be binding on any number of items (i.e. zero, one or more than one item);

(7) there is a cost per unit time incurred when the machine is operational, i.e., when any item is being set up or being produced.

The notation used in this paper are given below:

n : number of items.

T: cycle time in years.

D_i: demand rate for item i (i = 1, 2, ..., n), in units per year.

P_i: production rate for item i, in units per year.

$P_i^{\max}$: maximum production rate for item i, in units per year.

r_i:= D_i/P_i

$r_i^{\min}$:= $D_i/P_i^{\max}$. It is assumed that $\sum_{i=1}^{n} r_i^{\min} \le 1$. Otherwise, there would not be any feasible production schedule.

h_i: holding cost rate of item i in dollars per unit per year.

$H_i := \dfrac{D_i h_i}{2}$

S_i: shelf life of item i in years.

τ_i : setup time for item i in years.

c: cost per unit time of operating the machine (including setup times) in dollars per year.

A_i: setup cost for item i (excluding the machine operating cost during the setup.

The decision variables in this problem are the cycle time T, and r_i (i = 1, ..., n). Note that in Silver's (1995) approach, r_i is fixed to $r_i^{\min}$ for all items i except the item with the binding shelf life constraint. In this paper we attempt to find the optimal r_i for all the items as well as the optimal cycle time T. For a given $\mathbf{r} = (r_1, r_2, \ldots\ldots, r_n)$ and T, the total cost $Z(\mathbf{r}, T)$ comprises the machine setup cost, holding cost, and the machine operating cost. The resulting optimisation problem can then be formulated as given below.

Min $$Z(\mathbf{r}, T) = \frac{1}{T}\left(\sum_{i=1}^{n} A_i + c\sum_{i=1}^{n}\tau_i\right) + T\sum_{i=1}^{n} H_i (1 - r_i) + c\sum_{i=1}^{n} r_i \tag{1}$$

subject to

$$T \geq \frac{\sum_{i=1}^{n}\tau_i}{1-\sum_{i=1}^{n} r_i} \tag{2}$$

$$T \leq \frac{S_i}{1-r_i} \tag{3}$$

$$r_i \geq r_i^{\min} \quad \text{for all } i = 1, 2, \ldots\ldots, n \tag{4}$$

Constraint (2) is the constraint on the total available time in a cycle. The production time and setup time in a cycle can not exceed the cycle length. Constraint

(3) is the shelf life constraint. Constraint (4) ensures that the production rate for each item does not exceed the maximum possible rate.

3. Analysis of the model

For a fixed value of T,

$$Z(\mathbf{r}) = \frac{1}{T}\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} c\, r_i \tau_i + \sum_{i=1}^{n} \tau_i T H_i (1 - r_i)$$

$$= \frac{1}{T}\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} (c - H_i T)\, r_i \tau_i + \sum_{i=1}^{n} H_i \tau_i T \qquad (5)$$

As r_i increases (i.e. production rate of item i decreases), the operating cost of the machine increases by a factor of c. However, the average inventory level and hence the carrying cost decreases by a factor of H_i. Therefore, if the decrease in carrying cost is more than the increase in machine operating cost, it is best to decrease the production rate for the item with the highest value of H_i.

Let $i^* = \arg\min_i (c - H_i T)$. Note that i^* corresponds to the item with the highest value of H_i. If $(c - H_{i^*}T) < 0$, then $Z(\mathbf{r})$ is minimised by making r_{i^*} as large as possible, and making r_i, $i \neq i^*$ as small as possible. There fore, when $(c - H_{i^*}T) < 0$, for the problem with no shelf life constraint, the optimal value of r_i for a given value of T is

$$r_i = \begin{cases} r_i^{\min} & \text{for all } i \neq i^* \\ \dfrac{T(1) - \sum_{i \neq i^*} \sum_{i=1}^{n} \tau_i}{T} & \text{for } i = i^* \end{cases} \qquad (6)$$

However, when $(c - H_{i^*}T) \geq 0$, $Z(\mathbf{r})$ is optimised by having the lowest possible value of r_i for all the items. That is, for a fixed value of T, with $(c - H_{i^*}T) \geq 0$, and with no shelf life constraints, $r_i = r_i^{\min}$ for all i.

3.1. Optimal cycle time with shelf life restrictions

Let I_s be the set of items for which the shelf life constraint is binding. (For ease of exposition, we assume that $i^* \notin I_s$. However, the method proposed in this paper can be suitably modified when this assumption is violated). For $i \in I_s$, since the shelf life constraint is binding, the value of r_i should be such that $T = \frac{S_i}{1 - r_i}$. The optimal T and **r** = (r_1, r_2, ..., r_n) can now be determined. The optimal T has to be evaluated for two possible cases.

Case A: $(c - H_{i^*}T) \geq 0$

In this case, $$r_i = \begin{cases} r_i^{\min} & i \notin I_s \\ 1 - \dfrac{S_i}{T} & i \in I_s \end{cases} \tag{7}$$

Therefore,

$$\begin{aligned} Z(T, \mathbf{r}) = Z(T) &= \frac{1}{T}\sum_{i=1}^{n} A_i + \sum_{i \notin I_s} \left(\tau T H_i r_i^{\min} + c(1 - r_i^{\min})\right) \\ &\quad + \sum_{i \in I_s} \left(\tau T H_i \left(1 - \frac{S_i}{T}\right) + c\left(1 - 1 + \frac{S_i}{T}\right)\right) \\ &= \frac{1}{T}\left(\sum_{i=1}^{n} A_i + c\sum_{i \in I_s} S_i\right) + \tau T\left(\sum_{i \notin I_s} H_i r_i^{\min} + \sum_{i \in I_s} H_i\right) \\ &\quad + c\sum_{i \notin I_s}(1 - r_i^{\min}) - \tau\sum_{i \in I_s} S_i H_i \end{aligned} \tag{8}$$

where $|I_s|$ is the number of items for which the shelf life constraint is binding.

Therefore the objective now is to minimise $Z(T)$ given by (8) subject to constraints (2), (3) and (4). Constraint (3) need not be considered explicitly, since the production rate for all the items $i \in I_s$ have been adjusted to satisfy this constraint. Constraint (2) and (4) have to taken into account. Substituting (7) into (2), Constraint (2) can be rewritten as

$$T \le \frac{\sum_{i \in I_s} S_i - \sum_{i=1}^{n} \tau_i}{1 - \sum_{i \notin I_s} r_i^{\min} - |I_s|} \qquad (9)$$

Constraint (4) is satisfied by all $i \notin I_s$, since these are set equal to $r_i^{\min}$. Substituting the value of $r_i, i \in I_s$ from (7) into (4), constraint (4) can be rewritten as

$$T \ge \max_{i \in I_s} \left\{ \frac{S_i}{1 - r_i^{\min}} \right\} \qquad (10)$$

Let

$$T_a^* = \begin{cases} \sqrt{\dfrac{\sum_{i=1}^{n} (A_i + c\tau_i) - c \sum_{i \in I_s} S_i}{\sum_{i \notin I_s} H_i (1 - r_i^{\min})}} & \text{if } \left(\sum_{i=1}^{n} (A_i + c\tau_i) - c \sum_{i \in I_s} S_i \right) \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad (11)$$

$$T_1 = \max_{i \in I_s} \left\{ \frac{S_i}{1 - r_i^{\min}} \right\} \qquad (12)$$

and

$$T_2 = \frac{\sum_{i \in I_s} S_i - \sum_{i=1}^{n} \tau_i}{1 - \sum_{i \notin I_s} r_i^{\min} - |I_s|} \qquad (13)$$

Since $Z(T)$ given by (8) is a convex function of T, the value of T that minimises the total cost Z for Case A is

$$T_a = \begin{cases} T_1 & \text{if } T_a^* < T_1 \\ T_2 & \text{if } T_a^* > T_2 \\ T_a^* & \text{otherwise} \end{cases} \qquad (14)$$

The optimal cost Z_a^* corresponding to Case A can be determined by substituting for T from (14) into (8).

Case B: $(c - H_{i^*} T) < 0$

In this case,

$$r_i = \begin{cases} r_i^{\min} & i \notin I_s,\ i \neq i^* \\ 1 - \dfrac{S_i}{T} & i \in I_s \\ 1 - \sum\limits_{i \notin I_s, i \neq i^*} r_i^{\min} - |I_s| - \dfrac{\sum\limits_{i=1}^{n} \tau_i - \sum\limits_{i \in I_s} S_i}{T} & i = i^* \end{cases} \qquad (15)$$

Substituting (15) into (1), we get

$Z(\mathbf{r}, T) = Z(T) =$

$$\frac{1}{T}\sum_{i=1}^{n} A_i + \tau T \sum_{i \notin I_s, i \neq i^*} H_i r_i^{\min}\left(1 - r_i^{\min}\right) + c \sum_{i \notin I_s, i \neq i^*} r_i^{\min} + \sum_{i \in I_s} \left(1 - \frac{S_i}{T}\right)\left(1 - 1 + \frac{S_i}{T}\right)$$

$$+ c \sum_{i \in I_s} \left(1 - \frac{S_i}{T}\right) + T H_{i^*} \left(1 - \sum_{i \notin I_s, i \neq i^*} r_i^{\min} - |I_s| - \frac{\sum\limits_{i=1}^{n} \tau_i - \sum\limits_{i \in I_s} S_i}{T}\right)$$

$$+ c\left(1 - \sum_{i \notin I_s, i \neq i^*} r_i^{\min} - |I_s| - \frac{\sum\limits_{i=1}^{n} \tau_i - \sum\limits_{i \in I_s} S_i}{T}\right)$$

$$Z(T) = \frac{1}{T}\sum_{i=1}^{n} A_i + T H + K \qquad (16)$$

where $H = \sum_{i \in I_s} \ldots$, $K = \ldots \sum_{i=1}^{n} \tau_i \ldots$

Therefore the objective now is to minimise $Z(T)$ given by (16) subject to constraints (2), (3) and (4). Constraint (3) need not be considered explicitly, since the production rate for all the items $i \in I_s$ have been adjusted to satisfy this constraint. Constraint (2) is also implicitly satisfied, because of the way in which r_{i^*} is defined in (15). Therefore only constraint (4) has to be taken into account. Substituting (15) into (4), for $i \in I_s$ results in constraint (10), and substituting (15) into (4) for $i = i^*$ results in constraint (9). Therefore for Case B, the objective is to minimise $Z(T)$ given by (16) subject to constraints (9) and (10).

Let $$T_b* = \sqrt{\frac{\sum_{i=1}^{n} A_i}{\sum_{i \notin I_s, i \neq i*} H_i (1 - r_i^{\min}) + H_{i*} (\sum_{i \notin I_s, i \neq i*} r_i^{\min} + |I_s|)}}$$

(17)

Since $Z(T)$ given by (16) is a convex function of T, the value of T that minimises the total cost Z for Case B is

$$T_b = \begin{cases} T_1 & \text{if} \quad T_b* < T_1 \\ T_2 & \text{if} \quad T_b* > T_2 \\ T_b* & \text{otherwise} \end{cases} \tag{18}$$

The optimal cost Z_b* corresponding to Case B can be determined by substituting for T from (18) into (16).

4. Algorithm for Determining Optimal Cycle Time and Production Rate

Based on the results in the previous section, an algorithm can be developed for determining the optimal T and r_i, $i = 1, 2, ..., n$. For both the cases (Case A and Case B in section 3.1 above, determine the items whose shelf life constraints are binding. Determine the optimal T using (14) and (18) and the corresponding values of the objective function Z_{a*} and Z_{b*}. If $Z_{b*} < Z_{a*}$ then choose the optimal cycle time using (18) and r_i using (15); otherwise choose the optimal cycle time using (14) and r_i using (7). The algorithm for determining the optimal T and r_i is formally stated below.

1. Set $I_s = \{ \}$. Determine $i* = \arg\min_i (c / H_i)$.
2. Determine T_a and Z_a* using (14) and (8).
3. Check whether $T_a \leq \frac{S_i}{1 - r_i^{\min}}$ for all $i \notin I_s$. If not, include those items

i into the set I_s, and go to STEP 2. If the condition $T_a \leq \frac{S_i}{1-r_i^{\min}}$ is not violated for any $i \notin I_s$, then proceed to STEP 4.

4. Set $I_s = \{\ \}$.
5. Determine T_b and Z_b* using (18) and (16).
6. Check whether $T_b \leq \frac{S_i}{1-r_i^{\min}}$ for all $i \notin I_s$. If not, include those items i into the set I_s, and go to STEP 5. If the condition $T_b \leq \frac{S_i}{1-r_i^{\min}}$ is not violated for any $i \notin I_s$, then proceed to STEP 7.
7. If ($Z_{b*} < Z_{a*}$)
   Then the optimal $T = T_b$, and the optimal **r** can be determined using (15).
   Otherwise
   The optimal $T = T_a$, and the optimal **r** can be determined using (7).

Note that the optimal cycle time corresponding to Case B is never less than that corresponding to Case A. That is, $T_b \geq T_a$. A proof of this provided in the appendix.

We now show that the cycle time corresponding to the optimal cost will fall within the appropriate range. The application of the above algorithm can result in three possible situations:

i) $T_b \geq T_a \geq (c/H_{i*})$. In this situation, Case B is applicable. Therefore, the total cost Z is minimised by making r_{i*} as large as possible. Therefore $Z_{b*} \leq Z_{a*}$.

ii) $T_a \leq T_b \leq (c/H_{i*})$. In this situation, Case A is applicable. Therefore, the total cost Z is minimised by making all the r_i's as small as possible. Therefore $Z_{a*} \leq Z_{b*}$.

iii) $T_a \leq (c/H_{i*}) \leq T_b$. In this situation, both T_a and T_b fall within the appropriate range. Therefore, we can choose the cycle time that results in the lower cost.

In all the three situations above, the cycle time corresponding to the optimal cost falls within the appropriate range. That is $T_a \leq (c/H_{i*})$, if $Z_{a*} \leq Z_{b*}$ [situation (ii) or (iii)] and $T_b \geq (c/H_{i*})$, if $Z_{b*} \leq Z_a$ [situation (i) or (iii)].

5. Examples

We now illustrate the superiority of the proposed method using a few examples.

Example 1 : This example is the same as illustration 2 in Silver (1995). The data for the example is given below.

c = machine operating cost per year = \$1000.

item i	D_i	τ_i	A_i	h_i	P_i^{max}	S_i
1	1000	0.0005	70	10	3000	0.20
2	500	0.0010	80	12	2500	0.11
3	700	0.0015	135	15	2500	0.20

Table 1: Data for Example 1

Using the proposed algorithm, the following solution is obtained.

r_1 = 0.333, P_1 = 3000, r_2 = 0.305, P_2 = 1641.3623, r_3 = 0.280, P_3 = 2500,

T^* = 0.1582, Z^* = 4193.82.

In this case, both the proposed method as well as Silver's method obtain the same solution.

Example 2: The data for the example 2 is given below.

c = machine operating cost per year = \$100.

item i	D_i	τ_i	A_i	h_i	P_i^{max}	S_i
1	1000	0.0005	70	10	3000	0.20
2	500	0.0010	80	12	2500	0.11
3	700	0.0015	135	30	2500	0.20

Table 2: Data for Example 2

Using the proposed algorithm, the following solution is obtained.

P_1 = 3000, P_2 = 2500, P_3 = 1573.57, T^* = 0.1375, Z^* = 3762.56.

With Silver's (1995) method the following solution is obtained.

P_1 = 3000, P_2 = 1629.74, P_3 = 2500, T^* = 0.1587, Z^* = 3948.52.

For the same problem data given in Table 2, the solution obtained by Silver (1995) and the proposed algorithm for different values of the machine operating cost *c* is given below.

c	Cost of the Solution obtained by the Proposed Algorithm	Cost of the Solution obtained by Silver (1995)
0	3662.56	3853.98
100	3762.56	3948.52
500	4162.56	4312.71
1000	4662.56	4735.71

Table 3 : Cost of solution obtained by the proposed method and Silver for different values of *c*

Example 3: The data for the example 3 is given below.

c = machine operating cost per year = \$100.

item *i*	D_i	τ_i	A_i	h_i	$P_i^{\max}$	S_i
1	1000	0.0005	70	10	3000	0.09
2	500	0.0010	80	12	2500	0.11
3	700	0.0015	135	30	2500	0.20

Table 4: Data for Example 3

Using the proposed algorithm, the following solution is obtained.

P_1 = 2894.73, P_2 = 2500, P_3 = 1617.65, T^* = 0.1375, Z^* = 3771.73.

Silver's (1995) method cannot be used for this example, since more than one item (item 1 and item 2) have binding shelf life constraints.

As pointed out by the referee, the real payoff for the proposed method compared to Silver's (1995) method comes for situations where $(c - H_{i*}T) < 0$, which is unlikely to occur in practice often. However, the machine operating cost need not be very high in practice (since the model should consider only variable operating costs such as cost of consumables, spare parts, and electricity). Therefore, the case $(c\text{-}H_{i*}T) < 0$ is very much possible in practice depending on the problem parameters, although it might not occur often. It should also be pointed out that unlike earlier methods in the literature, the proposed algorithm can deal with cases where the shelf life constraint is binding for more than one item (see example 3).

6. Summary

This paper has proposed a new algorithm for determining the optimal cycle time and production rates for the family production scheduling problem with shelf life constraints. The proposed algorithm works even when there are more than one item with a binding shelf life constraint. The algorithm also determines the optimal production rate for all the items and is an improvement over Silver's (1995) method.

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Appendix

We show that when the algorithm for determining the optimal cycle time is used, the cycle time corresponding to Case B is never less than that corresponding to Case A. That is, $T_b \geq T_a$.

Proof:

Situation 1: We first prove the result for the situation when the set I_S remains the same for Case A and B. For a fixed value of **r**, the total cost *Z* is a convex function of *T*. In Case B, all the r_i's except for the item *i** has same value as in Case A. The value for r_{i*} is determined by (15), and is usually greater than $r_{i*}^{\min}$. If the r_{i*} determined by (15) is less than $r_{i*}^{\min}$, then constraint (9) is binding in both Case A and B, and therefore $T_b = T_a = T_2$. If $r_{i*} \geq r_{i*}^{\min}$ in Case B, then the coefficient of *T* in the cost function Z(*T*) is lesser in Case B, while the coefficient of (1/*T*) is the same for both. Therefore, $T_{b*} \geq T_{a*}$. The values T_1 and T_2 remain the same in both Case A and B, since the set I_S remains the same. Therefore, $T_b \geq T_a$.

Situation 2: We now prove the result when the set I_S is different for case A and B. Since $T_b \geq T_a$ for situation 1, the only possibility is that there are more items violating the shelf life constraint for case B (i.e. the set I_S corresponding to case A is the subset of the set I_S corresponding to case B). Let *k* be an item for which the shelf life constraint is binding in case B, but not in case A. Since the shelf constraint is not binding for item *k* in case A, from equation (3), and step (3) of the algorithm we get

$$T_a \leq \frac{S_k}{1 - r_k^{\min}} \tag{19}$$

Since the shelf constraint is binding for item *k* in case B, from (10) we get

$$T_b \geq \frac{S_k}{1 - r_k^{\min}} \tag{20}$$

Combining (19) and (20) gives the required result.