Division of Economics, EGC
School of Humanities and Social Sciences
Nanyang Technological University
14 Nanyang Drive
Singapore 637332

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Weihong HUANG and Wanying Wang

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Tel: +65 67905689
Email: D-EGC@ntu.edu.sg
http://egc.hss.ntu.edu.sg

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Weihong HUANG[a] and Wanying WANG[a*]

[a]Division of Economics, Nanyang Technological University, 14 Nanyang Avenue 637332, Singapore

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Abstract

Though the price–volume relations are widely documented by practitioners and empirical studies, few theoretical models can reproduce these relations and provide persuasive arguments. By simply generalizing the classical market maker framework, our heterogeneous agent model not only simulates satisfactorily the seemingly chaotic fluctuations in price and volume in a way that is highly compatible with the real market, but replicates patterns in the movements of price–volume, particularly those patterns used in technical analysis. Most importantly, based on this model, plausible economic arguments are provided to support the rationale of correlations between asset returns and volumes.

JEL classification: C63, G12, G17

Key Words: Price–volume relations, Heterogeneous beliefs, Technical analysis, Deterministic nonlinear dynamics

*Corresponding author. Tel: +65-90125091. Fax: +65-67955797. Email Addresses: wang0589@e.ntu.edu.sg(W. WANG), awhhuang@ntu.edu.sg(W. HUANG).

1 Introduction

Other than its simple interpretation as a liquidity proxy, speculators in real world have discovered the informational roles played by trading volume long time ago. For example, "It takes volume to make prices move," the famous Wall Street adage, reflects the positive correlation between volume and price movement. Fig. 1 shows a typical example of this positive relation, where the trading volume hit an unprecedented level while the price dropped down to a "bottom" in March 2009. Such a price–volume comovement pattern has appeared frequently and repeatedly in historical series in various financial markets (for example, Black Monday in 1987). Another widely accepted price-volume relation is "volume tends to be heavy in the bull market but light in the bear market", which indicates a positive correlation between the trading volume and the price change itself. Moreover, in the theory of technical analysis such as Murphy (1999) and Bulkowski (2000), volume is one of the most important references for the timing of entry and exit. Only if the chartists observe a sudden increased trading volume can they confirm the breakout of the trend-line.

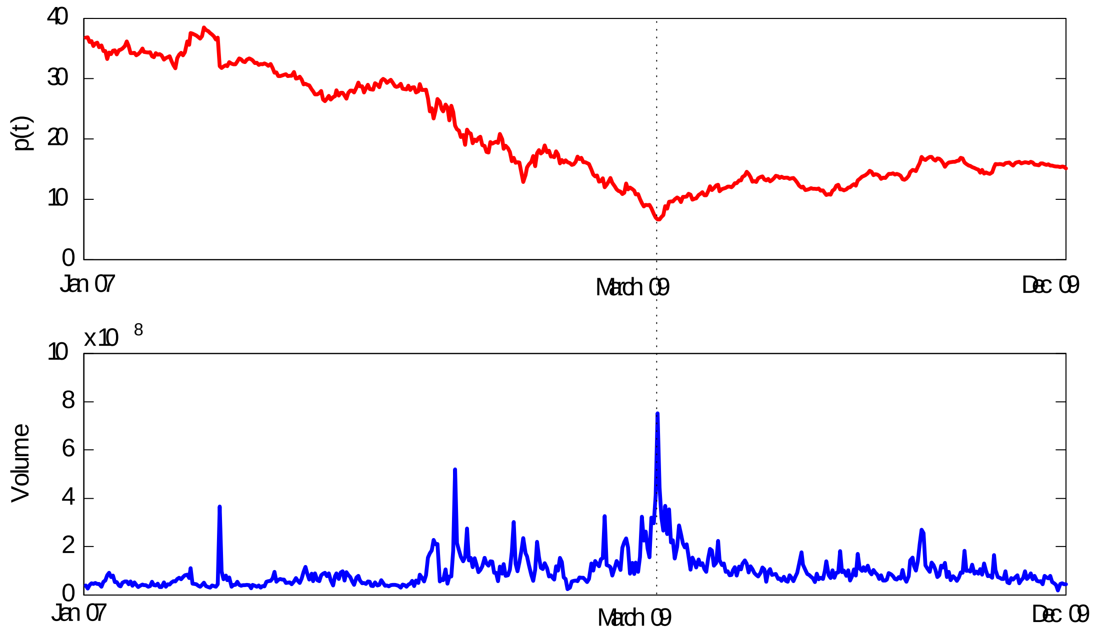

Fig. 1: Price–volume series of GE (Jan. 1st, 2007- Dec. 31st, 2009)

The present paper intends to model and justify the price–volume correlations from the economic perspectives of demand and supply. Following the market maker framework by Day and Huang (1990), two types of investors are examined: fundamentalists and chartists. The fundamentalists are informed traders and will buy/sell orders when the price is below/above the fundamental value. The chartists, on the other hand, are essentially trend followers. Either due to incomplete information or due to a simple belief that the history repeats itself, they make trading decisions basing on the investment values estimated from historical data. Interactions between these two types of traders, examined from a dynamic framework, enable us to generate price and volume associated simultaneously and to explore their correlations from economic point of view.

This paper distinguishes from existing literatures in the several aspects. Firstly, unlike all the previous theoretical studies, our model is purely deterministic instead of stochastic[1]. Although the

[1]Deterministic nonlinear dynamics have been proved to be very powerful on the simulation of stylized facts of financial markets. Twenty years ago, Day and Huang (1990) shows that the seemingly chaotic price fluctuations in

trading activities in reality are highly influenced by random factors and external shocks, focusing on the deterministic mechanism underlying makes it possible to explain the predictive power of charts from behavioral economics and justify why the same price-volume movements repeats itself all the time and everywhere. In fact, adding random errors on price and/or volume does not affect our analysis and conclusions but makes our simulations closer to the reality.

Secondly, in contrast to the most popular "sequential arrival of information" model (see Epp and Epps, 1976 and its followers), in which new information flows into the market and is disseminated to investors one at a time, our model does not impose any informational constraint.

Thirdly, the model is able to simulate all major features of the price-volume relations documented, which include

(i) a significant volume is accompanied by a large change in absolute value of price change;

(ii) the volume is heavy when the market is bullish and light when the market is bearish;

(iii) volume behaves in a specific pattern as an important signal in chart patterns.

The rest of the paper is organized as follows. In Section 2, a literature review related to current paper are provided. In Section 3, our model is investigated. Section 4 presents our simulation results, and Section 5 explains the theoretical implications of the results given by our joint dynamics. Section 6 is an extensive study on the positive relation of trading volume and price change *per se.* In Section 7, we test the existence of bi-directional nonlinear Granger causality between price and volume. The final section sets out the conclusion.

2 Literature review

Volume claims to be positively correlated with the asset prices through a large number of empirical evidences since 1987, the year that Black Monday happened [2]. Researchers have investigated

the stock market could be well-simulated by a deterministic nonlinear dynamics. After that Huang et al. (2010) and Huang and Zheng (2012) demonstrates that similar models can capture the features of all three types of financial crises. Huang and Wang (2011) also replicates all the chart patterns used in the technical analysis by a deterministic Heterogeneous Agent Model.

[2] Literature includes Crouch (1970a), Crouch (1970b), Epps (1975), Epps and Epps (1976), Morgan (1976), Westerfield (1977), Cornell (1981), Tauchen and Pitts (1983), Grammatikos and Saunders (1986), Harris and Gurel (1986),

market indexes and individual stocks, chosen different time intervals, selected different combinations of stocks, used data from different time periods in different common stock markets or different future markets, and finally confirmed the existence of a positive correlation between price and volume, and even stronger evidence for one between the trading volume and the absolute value of price change. Most importantly, Hiemstra and Jones (1994) proposes a nonparametric nonlinear Granger causality test and confirmed the nonlinear Granger causality in the price–volume relations in the U.S. market. The nonlinear causality relation between price and volume has subsequently been tested across different markets in numerous empirical studies[3]. Most of these studies are consistent with the hypothesis that asset returns and trading volumes have the nonlinear Granger causality relations. Therefore, as Lo et al. (2000) argues, "It is difficult to dispute the potential value of price/volume charts when confronted with the visual evidence".

Theoritically, motivated by Ying (1966), who states that "any model of the stock market which separates prices from volume or vice versa will inevitably yield incomplete if not erroneous results," a parallel theoretical study was developed to explain these relations in the 1980s and 1990s, especially after the Black Monday in 1987. The first possible explanation is a "sequential arrival of information" model, given by Copeland (1976), and later improved by Jennings and Barry (1983). This pattern of information arrival produces a sequence of momentary equilibria consisting of various stock price–volume combinations before the final complete information equilibrium is achieved. The second explanation for the relations is the "mixture of distributions hypothesis model" derived by Epps (1975), Epps and Epps (1976). Epps and Epps' model, similar to the sequential information arrival model, also builds a particular framework on the way speculators receive and respond to information to justify the positive relation between the volume and the price change *per se.* Some other possible models have also been established, including that of tax and non-tax related motives in Lakonishok and Smidt (1989) and the noise trader model in Long et al. (1990).

Gallant et al. (1992).

[3]e.g., Pisedtasalasai and Gunasekarage (2007) for South-East Asia emerging markets, Silvapulle and Choi (1999) for the South Korea stock market, Gĺźnduumlz and Hatemi-j (2005) for Central and Eastern European markets, and Saatcioglu and Starks (1998) for Latin America markets.

However,Previous theoritical literatures cannot explain the price–volume comovement well without any external conditions. For instance, the main critique against the sequential information model pointed out its prohibition on short sale. Both the "sequential arrival of information" models and the "mixture of distributions hypothesis" models require external constraints on information. Moreover, Banerjee and Kremer (2010) can only explain the positive price–volume correlations when investors have infrequent but major disagreements. In the recent ten years, the topic of price–volume is no longer fresh and attractive as before. But the puzzles of the price–volume relations are still unsolved.

On the other hand, the heterogeneous agent model (HAM), focusing on the interactions between different types of investors, became highly popular over the next two decades such as Beja and Goldman (1980), Day and Huang (1990), Chiarella (2003), David (2008) and Mendel and Shleifer (2012). However, most of the existing studies in heterogeneous beliefs focus on the price series, and little attention has been paid so far to the volume. One of the exceptions is Karpoff (1986). Heterogeneous agents with different personal valuations of the asset are introduced in the model and the simulation results are consistent with some established empirical findings including the positive correlation between the price and trading volume. Later on, Chen and Liao (2005) attempts to use an agent-based stock markets model to determine the price–volume series and reproduces the presence of the nonlinear Granger causality relation between the price and volume. They examines the dynamic relations of price–volume on both a macro and a micro level. The simulation results are mixed. Relations can be found in some results but not in others. Therefore, the conclusions of the simulation remaines inconclusive. Meanwhile, the authors could not explain the existence of the price–volume relations in their results by the generic property of the financial market itself.

To our knowledge, no existing literatures could replicate all the major features of price–volume movement and no studies could explain the mechanism behind the volume signals on price persuasively. This paper attempts to contribute to previous literatures by offering a new perspective with heterogeneous beliefs. It demonstrates that a purely deterministic nonlinear dynamics by modelling the interactions between different investors can just fill the gaps and uncover the underlying mechnism of price and volume dynamics.

3 Model

3.1 *Price dynamics*

Price dynamics are investigated under the classic framework of Day and Huang (1990). In this framework, six assumptions are needed for simplicity.

(i) There are only three types of traders in the market: the fundamentalist, the chartist, and the market maker. The fundamentalist and the chartist firstly trade with each other, and the market maker will take up the aggregate excess demand (or supply) and adjust the price in the next period accordingly.

(ii) The intrinsic properties of each group determine that investors cannot switch to the other group by their own willingness. For instance, the chartist cannot switch to the strategies the fundamentalist uses since the chartist is not able to obtain sufficient information to calculate the fundamental value.

(iii) The population for each group is the same, and no trader is allowed to enter or exit the market, so the group size remains unchanged for both the fundamentalist and the chartist for all the periods.

(iv) All the investors in each group are alike, which means that they share exactly the same strategy.

(v) Only one risky asset is available.

(vi) There is no budget constraint. Both types of traders can buy (or sell) as much as they are willing to based on their strategies.

3.1.1 *Fundamentalists*

The first type of player is the fundamentalist. Fundamentalists believe in the traditional fundamental approach of asset pricing and expect the asset price p_t will fluctuate in a reasonable zone (m, M) owing to some external disturbances but will eventually converge to its intrinsic value u_t^f[4]. By acknowledging information such as cash flows and dividends, fundamentalists could derive an accurate estimation of the fundamental value of the asset. In practice, they are assumed to be fund managers or professionals in banks or other financial institutions. They are confident and make their independent decisions without being influenced by trends or temporary fluctuations. From their point of view, any fluctuation is caused only by short-term external disturbances. Thus, they will decisively buy in when the fundamental value of the asset u_t^f is below the current price p_t. When the price lowers even further, the expected capital gains will be higher, so they will continue buying with even more orders. Similarly, when $u_t^f < p_t$, they will sell the orders, and the higher p_t is, the greater potential capital loss the fundamentalist will expect. Therefore, the excess demand of fundamentalists would be as:

$$\alpha(p_t) = \begin{cases} (u_t^f - p_t) \cdot A(u_t^f, p_t), & \text{if } m \leq p_t \leq M. \\ 0, & \text{if } p_t < m \text{ or } p_t > M. \end{cases} \tag{1}$$

Here, m and M are the minimum and maximum boundaries, respectively, of the price fluctuations set up by the fundamentalist. When the price exceeds the boundaries, the market will be considered to be unreasonable, so the fundamentalist will exit the market until the price returns to an efficient state. Besides, u_t^f is the fundamental value of the risky assets. In our model, what most distinguishes fundamentalists from chartists is that fundamentalists have sophisticated investment strategies and access to all internal information. Therefore, by updating the information for each

[4]Black (1986) has defined that an efficient market should be fluctuated within a reasonable bound $(\frac{1}{k}u, ku)$. Here, k is a pre-selected factor $(k > 1)$ and u is the fundamental value of the risky asset. By this definition, the assumed threshold set by the investors is flexible with respect to the change of u. In this paper, since the fundamental value u is no longer a constant, but an endogenous determined variable, m and M will also be changed in each period.

period Ω_t, they are able to obtain the fundamental value of the asset for each period u_t^f. Here, $A(\cdot)$ is a positive function with respect to the fundamental value u_t^f and the current price p_t. It depicts the psychological behaviors of investors so that, when the price moves close to the topping price M, the probability of losing the existing gains increases. When the price moves close to the bottoming price m, the probability of missing the opportunity to buy the stock is also higher. Details about this function can be found in the Appendix.

Particularly, In other studies such as Day and Huang (1990), the fundamental value u_t^f is treated as a constant. However, in practice, it should be noticed that stock prices have a tendency to rise anyway in the long run. Therefore, to capture the general upward trend in the price series, the present paper will reasonably assume that the fundamental value u_t^f increases with business cycles. It is assumed that the length of each business cycle is $n \cdot S$ periods ($n \geq 2$). The business cycle consists of an expansion for $(n-1) \cdot S$ periods with the economic growth rate of g and a following recession for S period with a growth rate of $(-g/2)$.

The economic growth rate $g(t)$ is

$$g(t) = \begin{cases} g, & t \in [(n(i-1) \cdot s, (n \cdot i - 1)s) \\ -g/2, & t \in [(n \cdot i - 1)s, n \cdot i \cdot s] \end{cases}, \ i = 1, 2, 3, ..., n. \tag{2}$$

Therefore, the fundamental value u_t^f increases steadily with the dynamics as

$$u_{t+1}^f = (1 + g(t)) \cdot u_t^f. \tag{3}$$

3.1.2 *Chartists*

The other type of player is the chartist, also known as the "noise trader" in Black (1986). Chartists are assumed to have weaker positions in the market. They do not concern themselves with the reasons behind the fluctuation and simply observe the assets from the market itself. They normally need to extract two signals from the market, the first one is the trend.

One of the chartists' most important beliefs is that prices move in trends in a certain psychologically comfortable zone $[\mathbf{P}_{k-1}, \mathbf{P}_k)$. Most of the techniques used by chartists in fact simply chase the price up and down, implying that their jobs are to identify the existing trend and assume that the current price p_t will keep moving in this trend. In this way, our paper simplifies the chartists as the trend-followers. When the price p_t is above their expected short-run investment value u_t^c, the existing trend is upward so the higher the asset pricing, the more enthusiasm chartists have to buy the stocks. Moreover, when p_t is below u_t^c, the trend is to sell and they will sell even more when the degrees of downward trends are large. Thus, the psychological thresholds $\mathbf{P}i$ $(i = 1, 2, 3, \ldots, n)$ are defined as the resistance and support levels in technical analysis[5] by dividing the wholly trading regime $[\mathbf{P}o, \mathbf{P}n]$ into n mutually exclusive sub-regimes with the same length according to their previous trading experiences, that is

$$\mathbb{P} = [\mathbf{P}o, \mathbf{P}_1) \cup [\mathbf{P}_1, \mathbf{P}_2) \cup ... \ \cup [\mathbf{P}_{n-1}, \mathbf{P}_n], \tag{4}$$

where $\mathbf{P}_{k-1} - \mathbf{P}_k = \lambda$, which is a constant.[6]

The simplest expression for the excess demand for chartists $\beta(p_t)$ can be defined as a linear function of the spread between p_t and u_t^c, which is

$$\beta(p_t) = b \cdot (p_t - u_t^c), \text{ if } p_t \in [\mathbf{P}_{k-1}, \mathbf{P}_k), \tag{5}$$

where $b > 0$ is constant and measures the strength of chartists' responses to the price deviation.

The second essential information that chartists want to observe from the market is the short-run investment value u_t^c. Unlike the fundamentalist, the chartist is either not able to access the internal information, or simply does not care about such exclusive news. Chartists will extrapolate their

[5] The resistance level and support level have been well-documented in books introducing the technical analysis, such as Murphy (1999). They are self-explanatory, indicating that when p_t gets closer to $\mathbf{P}_{k-1}$, a sudden buying interest is substantially increased to overcome the selling pressure and normally the decline will be easily halted at this level. On the other hand, p_t around the resistance level $\mathbf{P}_k$ implies the belief that the sudden increased selling pressure outweighs the original buying pressure and a price advance is turned back.

[6] If we assume $\mathbf{P}o = 0$, the trading regime therefore is $\mathbb{P} = [0, \lambda) \cup [\lambda, 2\lambda) \cup ... \cup [(n-1)\lambda, n\lambda]$.

expectations of the investment value based on the prices in previous periods. For example, it is well known that most technicians will use technical indicators calculated by past prices to assist them to estimate future prices, such as oscillators or moving averages. However, the present paper adopts an adaptive belief of u_t^c following Huang and Wang (2011) and Huang and Zheng (2012). The mechanism is as below.

Assume at period one the initial price p_t locates at the kth regime $[\mathbf{P}_{k-1}, \mathbf{P}_k)$. The short-run investment value can simply equal the average of the top and the bottom threshold prices

$$u_t^c = (\mathbf{P}_{k-1} + \mathbf{P}_k)/2, \text{ if } p_t \in [\mathbf{P}_{k-1}, \mathbf{P}_k), k = 1, 2, ..., n. \tag{6}$$

After one-step price dynamics in period two, there are two possibilities.

Case I If the current price p_t decreases to p_{t+1} insignificantly, remaining at the same regime, there are sufficient reasons for the chartist to believe that the short-run investment value remains the same, that is,

$$u_{t+1}^c = u_t^c = (\mathbf{P}_{k-1} + \mathbf{P}_k)/2, \text{ if } p_t \in [\mathbf{P}_{k-1}, \mathbf{P}_k), k = 1, 2, ..., n. \tag{7}$$

Case II When the price in the current period p_t escapes from the original regime $[\mathbf{P}_{k-1}, \mathbf{P}_k)$ to a lower regime $[\mathbf{P}_{k-2}, \mathbf{P}_{k-1})$ or some even lower regimes, breaking the support level, the "regime switching" appears[7]. Under this condition, the chartist will expect that it is not simply the regular fluctuations but the change in the short-run fundamental value for the specific stock that leads to the jump in price.

$$u_{t+1}^c < u_t^c \text{ and } u_{t+1}^c = (\mathbf{P}_{k-2} + \mathbf{P}_{k-1})/2, \text{if } p_t \in [\mathbf{P}_{k-2}, \mathbf{P}_{k-1}), k = 1, 2, ..., n. \tag{8}$$

Following Huang et al. (2010), for each period t, the short-run investment value can be calculated

[7] Similar regime switching processes in nonlinear investment strategies have been widely applied in previous literatures, such as Huang and Day (1993), Day (1994), Ang and Bekaert (2002), Guidolin and Timmermann (2007), Guidolin and Timmermann (2008).

as

$$u_t^c = (\lfloor p_t/\lambda \rfloor + \lceil p_t/\lambda \rceil) \cdot \lambda/2, \text{ if } p_t \in [\mathbf{P}_{k-1}, \mathbf{P}_k), k = 1, 2, ..., n. \tag{9}$$

Thus, the chartist can substitute their expectations on the short-run investment value into Eq. (5) and form their excess demand (or supply) for the risky asset.

3.1.3 *The market maker*

Following Day and Huang (1990), to absorb the excess demand from the two agents mentioned above, a third agent, the market maker is introduced. It is not uncommon to see the existence of the dealers in a lot of stock exchanges such as NYSE and Nasdaq Stock Exchange. Market makers can help accelerate the liquidity of market when most of the investors in the markets hold the identical expectation of the price trend and enable the market participants who concern the cost of trading enter or exit a desired position in a very short period of time. Besides, another advantages of market maker framework is that, in our model, market makers make nontrival trading volume possible even when fundamentalists and chartists share the same prediction of future price movement. Consequently, we can avoid discussing the problems in Berrada et al. (2007). Meanwhile, with the introduction of market maker, this paper can also explain the phenomenon that liquidity and volume seem unrelated over time.

The mechnism for the market maker is to adjust the asset price in the next period according to the current aggregate demand from the fundamentalists and chartists. To balance their inventories, the price would be adjusted to increase following net buy orders, and the price would decline when holding net sell orders from investors. The price dynamics are therefore completed as a one-dimensional nonlinear process

$$p_{t+1} = p_t + \eta \cdot (\alpha(p_t) + \beta(p_t)). \tag{10}$$

In Eq. (10), η is the speed of adjustment, the measure of the adjustment speed of market maker according to the excess demand.

3.2 *Volume dynamics*

To give a plausible definition of volume under the framework of market maker, two situations in the market need to be investigated. In the first case, fundamentalists and chartists both share the same forecasting opinions about the future trend of the asset prices ($\alpha(p_t) \cdot \beta(p_t) > 0$). Both will trade with the market maker so the trading volume for the stock is the absolute value of the aggregate excess demand.

However, according to Black (1986), information traders or fundamentalists will mostly trade with the noise traders or the chartists, which means that they will hold opposing opinions concerning the future price of the asset ($\alpha(p_t) \cdot \beta(p_t) < 0$). Therefore, in the second case, fundamentalists and chartists will trade with each other first and the market maker will take up the remainder of the excess demand to obtain a liquid market. The volume in this case is equal to the maximum of the absolute value of the excess demand for each group.

In summary, volume can be defined as

$$V_t(p_t) = \begin{cases} \left|\alpha(p_t) + \beta(p_t)\right|, & \text{if } \alpha(p_t) \cdot \beta(p_t) > 0, \\ \max(\left|\alpha(p_t)\right|, \left|\beta(p_t)\right|), & \text{if } \alpha(p_t) \cdot \beta(p_t) < 0. \end{cases} \tag{11}$$

4 Model simulations

The price–volume signal is one of the tools widely adopted by technicians. By identifying specific signals to confirm the future trend, the technicians are able to determine the selling and buying signals. In this section, the simulation results (the price series in the top panel and the volume series in the bottom panel) will be offered to replicate various price–volume relations. To demonstrate both the capability and the robustness of our model, more importantly, to preserve continuity and unity, we shall adopt a default parameter set ($u_1^f = 50$, $d_1 = d_2 = -0.3$, $k = 2$,

$\lambda = 7.5,\ s = 25,\ a = 1,\ b = 2.25,\ \eta = 1,\ g = 0.0008,\ n = 4$) through the whole paper so that different simulations differ only on the initial price p_0.

4.1 *Visual price–volume comovements*

The most famous example of the positive relation between the absolute change of price and the magnitude of trading volume is the Black Monday of 1987 in the U.S. market. Fig. 2 provides a typical price–volume series generated from the default parameter set with $p_0 = 61.69$. In this figure, when the volume hits a certain warning line of volume $V_t(p) = 10$, the corresponding asset price is either in the peak or in the bottom periods.

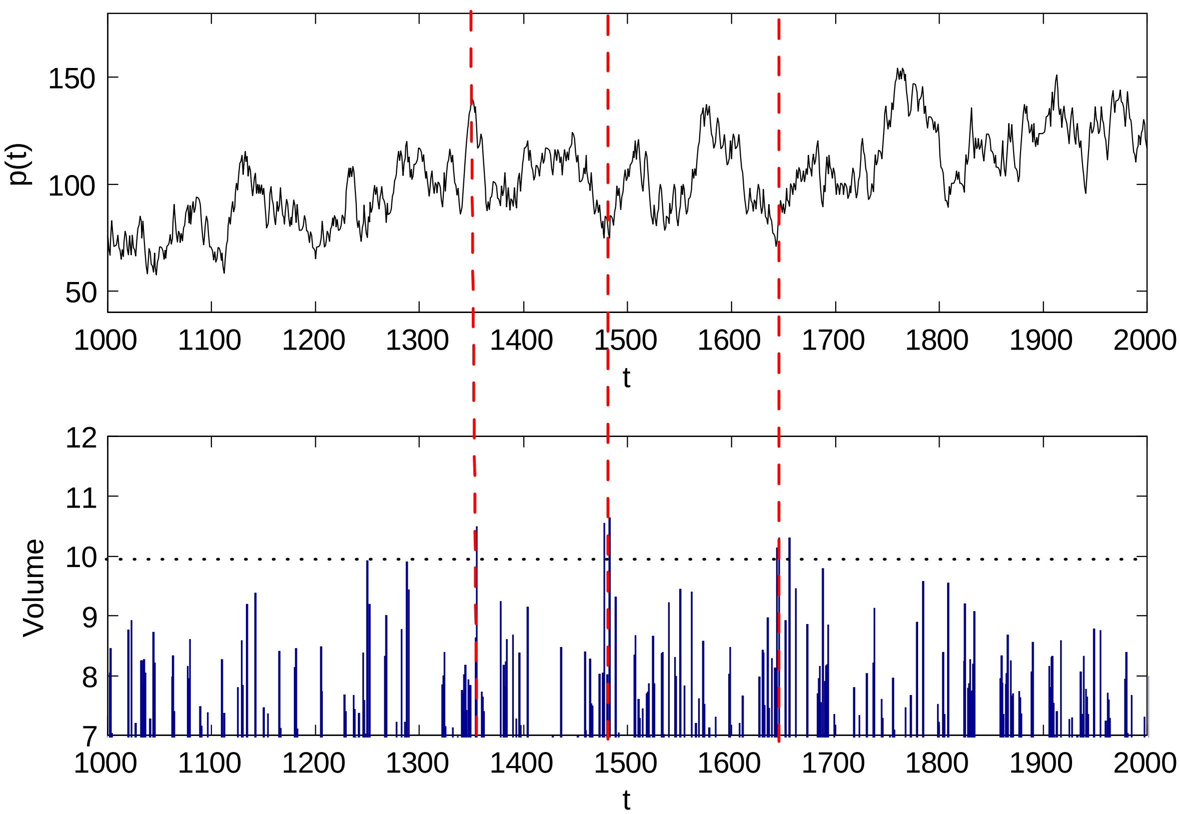

Fig. 2: Positive correlation between price and volume: the dashed lines link the peaks or the

bottoms in the price series to their corresponding volumes. The dotted line in the bottom panel indicates the volume level that $V = 10$.

4.2 *Informational role of trading volume on asset returns*

Another illustration of the price–volume comovements is given in Fig. 3. In all the other periods, the volumes are around the level $V_t = 5$. However, when $t = 139, V_{139} = 19.22$, which is almost four times the daily average volume. Meanwhile, the corresponding price $p_{139} = 27.17$ also drops below the resistance line $p = 40$. This again confirms the price–volume comovements observed by practitioners in daily stock markets and confirms in particular the importance of volume as a tool to determine the break of the resistance line in technical analysis. Furthermore, Fig. 4 shows the relation between the asset returns ($r_t = (p_t - p_{t-1})/p_{t-1}$) and the trading volume. The series of asset returns apparently fluctuates around $r = 0$ randomly. In the period $t = 140$, r_{140} is equal to 0.71, an unprecedented high level as well, which means that the trading volume V_{139} does have the predicting power on the asset return r_{140}. Therefore, this example successfully demonstrates that our model is able to replicate both the visual comovements between price and volume and the quantitative predicting power from volume to the returns.

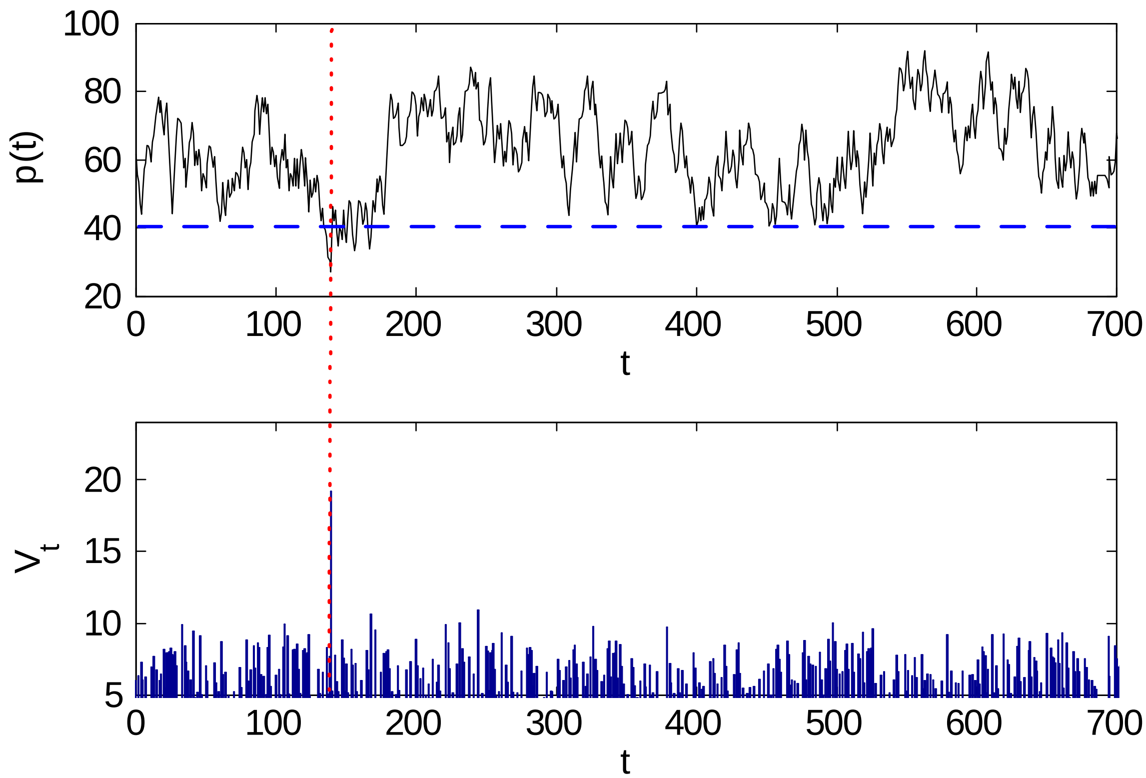

Fig. 3: The volume signal on the breakout of resistance line: the dotted line links the breakout in the price series in the top panel to the corresponding volume in the bottom panel and the dashed line in the top panel indicates a resistance line based on long-term observation. When the volume hits to a significant high level, the corresponding price series in the top panel also breaks the resistance line.

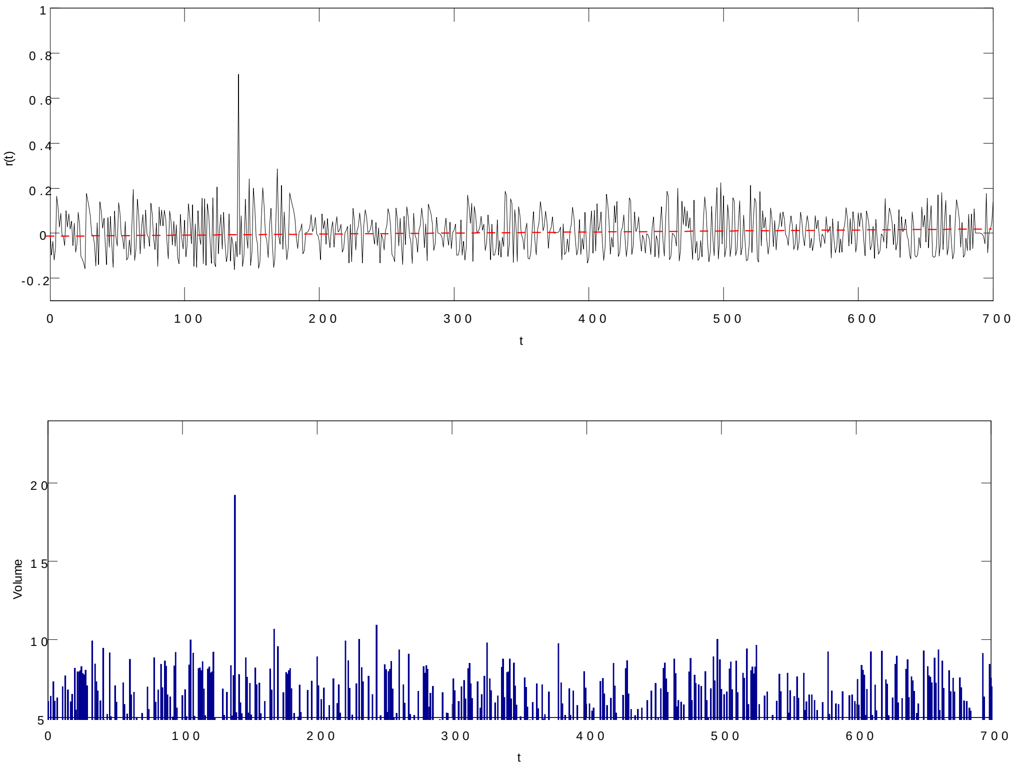

Fig. 4: Corresponding relations between the asset price return and trading volume: the top panel demonstrates the corresponding returns of the price series in Fig. 3. The dashed line in the top panel indicates that $r_t = 0$.

4.3 *Price–volume signals in chart patterns*

Chart patterns such as the head and shoulders and the double tops have been widely used by practitioners in the last century. However, because the identification of chart patterns is a purely visual-aid decision-making process, it has attracted criticisms from academics. Even the technicians themselves admit that not all the patterns work in the real market. As a result, technicians rely heavily on the volume signal as the most essential criterion to help them to decide whether the patterns fail or complete successfully.

Although different technicians have different perceptions, the major signals of volume include:

- The corresponding volume of each peak during the patterns should be significant. This property is compatible with the price–volume comovements we showed above.
- The general trend of volume during the chart pattern should be downward, which indicates that the selling power (or the buying power) is weakening during the pattern.
- The volume is substantially increased when the price breaks the trend-line (such as the neck-line in the head and shoulders bottom pattern). This property is the most meaningful signal required by technicians from the observation of trading volume. The most frequently asked questions of technicians include: When the head and shoulders bottom pattern appears to be complete? Will the price drop back below the trend-line again? Sometimes, the buying pressure is not sufficiently large to support a price increase above the trend-line. Therefore, only when the corresponding volume is substantially increased can the technicians confirm that the demand for the stock is very significant this time, which implies the validity of this breakout.

The following section demonstrates that our model is able to replicate this volume signal.

4.3.1 *Volume signals in the head and shoulders pattern*

In the time series shown in Fig. 5, a head and shoulders bottom pattern appears in the simulations when $p_0 = 61.69$ from $t = 700$ to $t = 800$. Bottoms appear including the left shoulder, the head and the right shoulder, from left to right. The volume series indicates the validity of the bottom as usual. The corresponding volume is significantly high in each bottom, and the volume trend is generally downward until the breakout. At the end of the bottom, the suddenly increased volume confirms the breakout of the neckline, that is, the imaginary line connecting the two rises between the shoulders and the head.

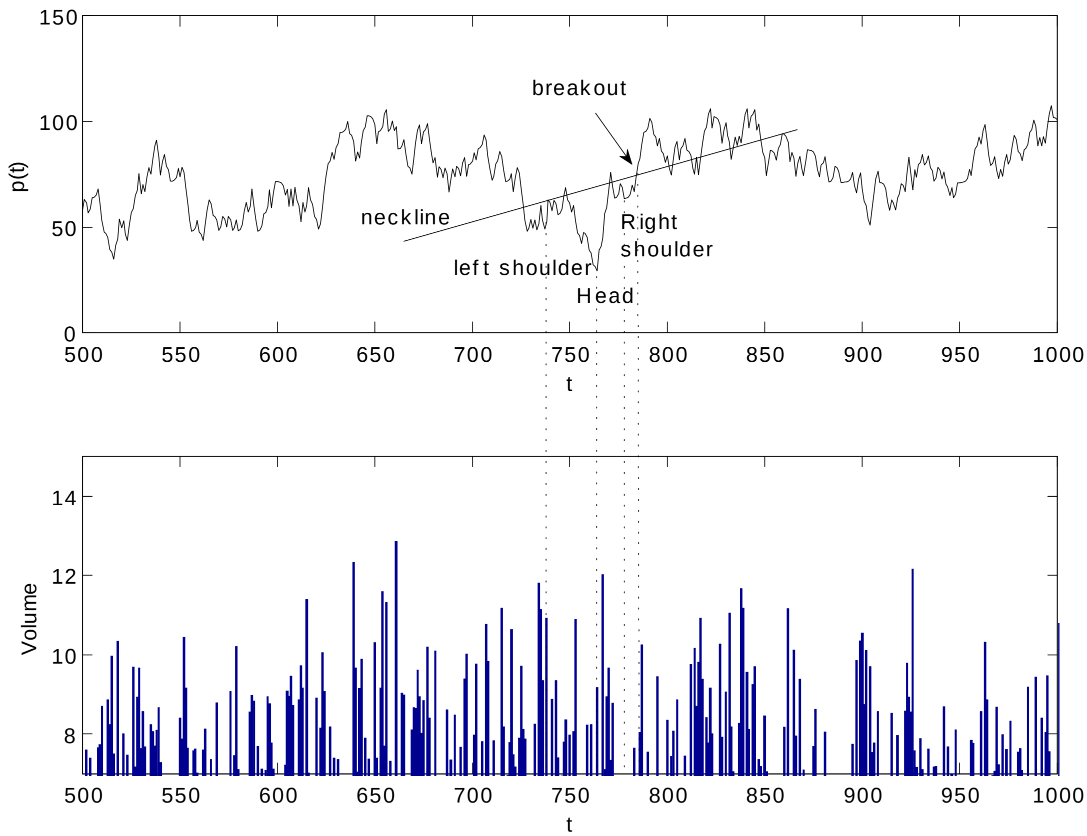

Fig. 5: Price–volume signals in the head and shoulder pattern: the dotted lines link each peak in the price series in the top panel to the corresponding volume in the bottom panel.

4.3.2 *Volume signals in the double bottoms and the double tops*

Fig. 6 shows our simulations of the double bottoms pattern closely followed by a double tops when $p_0 = 61.62$. In both patterns, the trading volume of each peak is comparatively heavy. Among them, the highest volume occurs on the left bottom (top) and diminished volume appears on the right bottom (top). The volume trend is generally downward. Finally but most importantly, the breakout volumes in both patterns are very heavy.

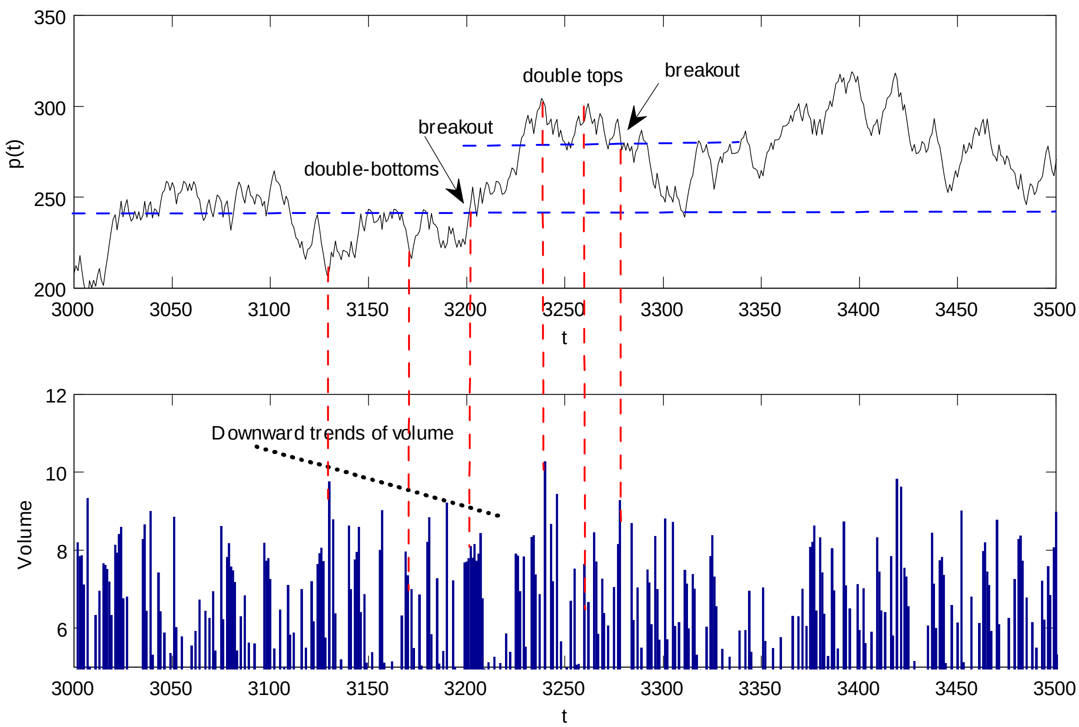

Fig. 6: Price–volume signals in the double tops and double bottoms patterns: In the top panel, a double-bottoms followed by a double-tops is presented. The horizontal dashed lines are the resistence/support lines. Besides, the vertical dashed lines connect the peaks in the price series to their corresponding volume in the bottom panel.

5 Theoretical implications

One of the advantages of using the HAM to simulate price and trading volume series is that we can determine both series simultaneously with the same simple market mechanism, the buying and selling pressures. Indeed, from the perspective of technicians, the price is not determined by its own value but by demand and supply from the market. Meanwhile, the trading volume is also closely determined by the demand and supply. Therefore, under the framework of our nonlinear chaotic model, the price and volume is undoubtedly nonlinearly correlated in some way. This section explains the internal mechanisms between trading volume and the asset returns, such as the phenomenon in the financial market mentioned in Fig. 4 in section 4.2.

For a plausible explanation for the informational role of volume, two conditions in Eq. (11) are examined separately.

Case 1 $V_t(p_t) = |\alpha(p_t) + \beta(p_t)|$ *(*$\alpha(p_t) \cdot \beta(p_t) > 0$*)*

It is the case that both the fundamentalist and the chartist agree on the future trend and either buy or sell the orders together Following from Eq. (10), the absolute price change thus can be simplified to $|r_{t+1}| = |p_{t+1} - p_t| = |c \cdot (\alpha(p_t) + \beta(p_t))| = c \cdot V_t(p_t)$. In other words, the absolute value of price change (or the asset returns) $|r_{t+1}|$ is proportional to the volume $V_t(p_t)$.

Case 2 $V_t(p_t) = \max(|\alpha(p_t)|, |\beta(p_t)|)$ *(*$\alpha(p) \cdot \beta(p) < 0$*)*

It is the case that the fundamentalist and the chartist disagree on the future trend of the price movement. In this case, a significant rise in volume would signal a large magnitude of either $|\alpha(p_t)|$ or $|\beta(p_t)|$. Based on Eq. (1) and Eq. (5), the divergence $|p_t - u_t^c|$ or $\left|p_t - u_t^f\right|$ is very large.[8]

[8] To explain this causality, we should understand that, in the second condition, when $|\beta(p_t)|$ is very large, according to Eq. (5), $|p_t - u_t^c| = |\beta(p_t)|/b$ is also very large. However, the first condition, when $|\alpha(p_t)|$ is very large, implies that either the chance function $A(\cdot)$ (see Appendix) or the divergence of current price p_t from the fundamental value u_t^f, or even both, are very large. In the other word, $A(\cdot)$ is also positively correlated to the divergence $\left|p_t - u_t^f\right|$. Therefore, we can conclude that, in both conditions, a large magnitude of the excess demand of each group implies a large divergence of $\left|p_t - u_t^f\right|$ or $|p_t - u_t^c|$.

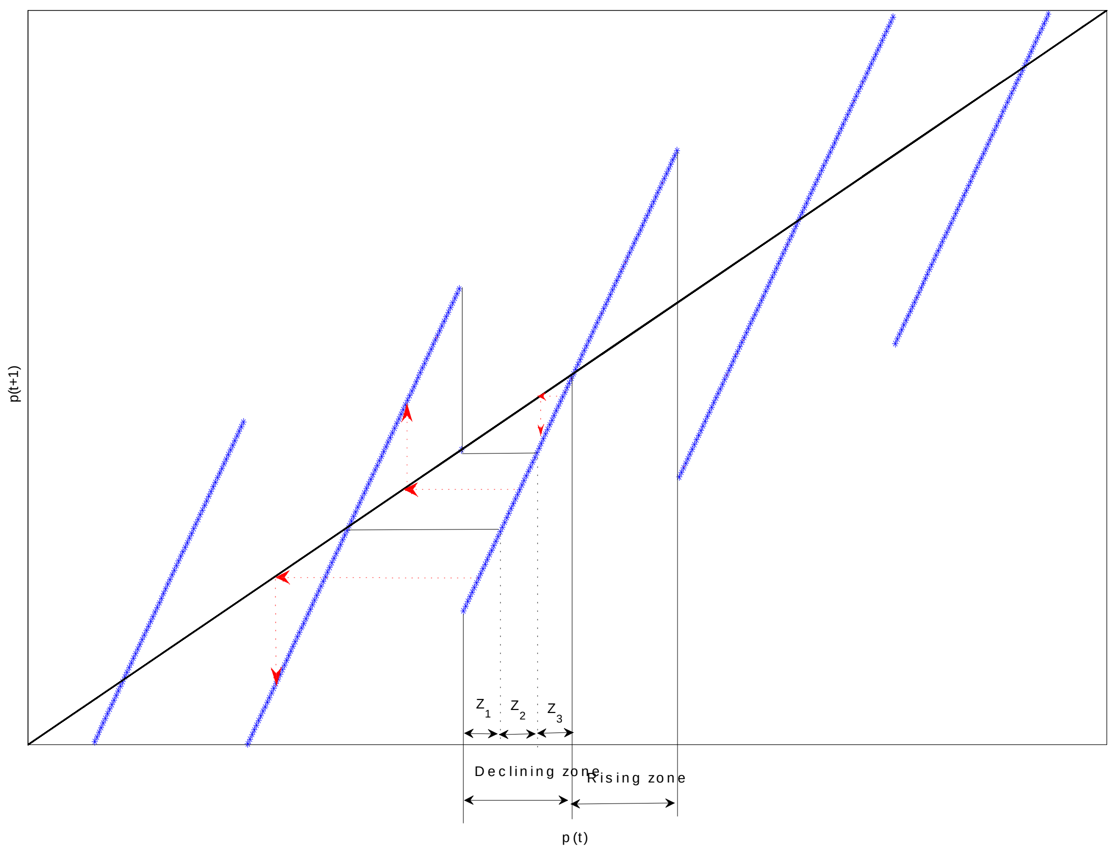

Fig. 7: Illustration of sudden declining zone, disturbing declining zone and the smooth declining zone when the market cosists only of chartists. The figure is the phase diagram that depicts the nonlinear dynamics in the price series. With differing distances to the equilibrium $p_t = u_t^c$, three different declining zones can be classified as Z_1, Z_2, Z_3.

To further explain this case, a phase diagram of our nonlinear price dynamics is provided in Fig. 7. In our deterministic model, the future price movement p_{t+1}, p_{t+2} is determined only by the current price p_t. For example, if one step-wise dynamics is considered, when p_t falls above the 45° line, p_{t+1} will rise, and when p_t is below the 45° line, the price will decline accordingly. In this way, each exclusive sub-regime $[\mathbf{P}_{k-1}, \mathbf{P}_k)$ in the figure can be divided into two different zones: the rising zone and the declining zone. If we take the simplest scenario, that the market consists only of the chartists, as an example, the equilibrium $p_{t-1} = p_t$ (which is the intercept of the price dynamics and the 45° line) exists when $p_t = u_t^c$. The interval $(u_t^c, \mathbf{P}_{k+1})$ is the rising zone, and the interval $(\mathbf{P}_k, u_t^c)$ is the declining zone. Similarly, if two step-wise dynamics are considered in the declining zone, with different distances to the equilibrium, three different declines can be further classified as "the smooth decline," "the disturbing decline," and "the sudden decline."

The smooth declining zone, depicted as Z_3 in the figure, is very close to the equilibrium (which implies $|p_t - u_t^c|$ is very small). When p_t falls into Z_3, the one step-wise price p_{t+1} will decline and remains in the declining zones in the same regime, which implies that p_{t+2} will also decline.

The sudden declining zone, depicted as Z_1, is located around the bottom of the price regime (which implies $|p_t - u_t^c|$ is very large). When p_t falls into Z_1, the one step-wise will shift to the declining zone in lower regimes, which implies that p_{t+2} will also decline. In this scenario, the price will drop significantly across different regimes.

The disturbing declining zone, denoted Z_2, is the period between the smooth declining zone and the sudden declining zone. When p_t falls into Z_2, the one step-wise price p_{t+1} will decline to the rising zone of the lower regime, which implies that p_{t+2} will instead rise.

By assuming that the difference between u_t^c and u_t^f is not dramatic, it is reasonable to believe that the current price falls into the suddenly declining zone, by the implication of the large divergence between p_t and u_t^c (or u_t^f). Therefore, in both cases, our deterministic nonlinear dynamics with heterogeneous beliefs is able to provide sufficient evidence of the commonly seen price–volume comovements.

6 Higher (lower) volume in the bull (bear) market

Another well-known saying regarding volume is, "the volume tends to be higher in the bull market and lower in the bear market." This relation between the trading volume and the price change *per se* is also widely accepted. Karpoff (1987) makes the point that it is not inconsistent that volume may correlate positively with both the absolute change of price or the price change *per* *se.* "It is likely that the V, Δp relation is not monotonic and the V, $|\Delta p|$ relation is not a one-one function."

In Section 5, to support the hypothesis that the interactions between heterogeneous agents may be the mechanism that jointly determines the price and volume series, we aim to establish a model that is as simple as possible and assumes that the population of each group of investors remains unchanged for all the periods. If we allow the group size of investors to change from period to period, it can be verified that the volume is heavier (lighter) in bull (bear) market without any further assumptions.

The population of investors in a certain market is never unchanging. For private financing purposes, most ordinary households are also willing to speculate in the market, but they are not well trained and have no sophisticated strategies. As a result, they can only observe the market, entering the market when it is bullish to chase the trend, and exiting when it is bearish. On the other hand, fundamentalists, as professional fund managers, will not be influenced by the market fluctuations, and so their population remains the same. Therefore, the population ratio of chartists to fundamentalists, denoted D, is higher in a bull market and lower in a bear market. We assume D to be a function of p_t only. In the decision-making process of chartists, when the price enters a new higher regime, not only is the short-run investment value u_t^c updated to a higher value, but the population also increases by a factor d, that is, $D(p_t) = 1 + d \cdot \lfloor p_t/\lambda \rfloor$. $d(d > 0)$ is a constant that measures the sensitivity of noise traders' willingness to enter the market based on the market price.[9] The excess demand from chartists is modified as $\beta(p_t) = b \cdot D \cdot (p_t - u_t^c)$.

To test whether the improvement in the model can help us to represent the relation between the trading volume and the bull (bear) market, we select 100 initial values $p_1 \sim N(61.65, 0.05)$ randomly, and examine these 100 price–volume series in Table 1.

Firstly, the average volume for upticks and the average volume for downticks are calculated. The results show that, even under the original assumption that $d = 0$, our model is able to capture the

[9] This implies that, if we assume the populations of the chartists and the fundamentalists are the same in the initial regime $(0, \lambda]$, in the following regime $(\lambda, 2\lambda]$, the population ratio of chartists to fundamentalists D is then $(1 + d)$. In this way, in the kth regime $((k-1)\lambda, k\lambda]$, the ratio D is $(1 + (k-1)d)$.

Table 1: Trading Volume in Bull and Bear Markets
Firstly, under different population ratios of heterogeneous agents (denoted by d), we apply Monte Carlo tests by generating 100 initial prices randomly distributed by $N(61.65, 0.05)$. The average volume for upticks and the average volume for downticks in each series are then calculated, and the mean of the average volumes in these 100 series is also obtained. Secondly, we apply the rank-sum test to examine whether the difference between these two means is significant.

d	$E(V_t)$ when $\Delta p_t > 0$	$E(V_t)$ when $\Delta p_t < 0$	p-value for rank-sum test
d=0	2.5638	2.4998	0.0000***
d=0.01	3.6608	3.5760	0.0000***
d=0.05	9.7150	9.6082	0.0108***

generic property of the financial market, that the average trading volume for upticks significantly exceeds that for downticks, which is in accordance with the phenomenon we observe in the market. If the parameter D is introduced, the differences between the two indicators become even larger. Secondly, to see whether the average volumes of the two groups differ significantly, following Epps (1975), we apply the Wilcoxon rank-sum test. The result clearly rejects the null hypothesis that the difference between the medians is equal to zero (the median difference is more significant when $d = 0$).

7 Nonlinear Granger causality test of the price–volume relation

Empirical studies failed to discover price–volume relations until Hiemstra and Jones (1994), which provided a nonlinear Granger causality test to investigate nonlinear relations between asset prices and trading volume. Since then, the nonlinear relation between price and volume has been proven in a variety of markets and countries. Therefore, to further examine the validity of our model, this paper uses the nonlinear Granger causality test to check whether there is significant evidence in our simulations for the existence of nonlinear price–volume relations.

7.1 *Nonlinear Granger causality test*

The nonlinear Granger causality test given by Hiemstra and Jones (1994) uses a nonparametric statistical method to uncover price–volume relations.

Consider two strictly stationary and weakly dependent time series $\{X_t\}$, $\{Y_t\}, t = 1, 2, 3, ...,$ where X_t^m is the m-length lead vector of X_t , and $X_{t-L_x}^{L_x}$ and $Y_{t-L_y}^{L_y}$ are the Lx-length and Ly-length lag vectors of X_t and Y_t respectively.

For given values of m, L_x, and $L_y \geq 1$, $e > 0$, Y does not strictly Granger-cause X if

$$\begin{aligned} &\Pr(\|X_t^m - X_s^m\| < e \mid \left\|X_{t-L_x}^{L_x} - X_{s-L_x}^{L_x}\right\| < e, \left\|X_{t-L_y}^{L_y} - X_{s-L_y}^{L_y}\right\| < e) \\ &= \Pr(\|X_t^m - X_s^m\| < e \mid \left\|X_{t-L_x}^{L_x} - X_{s-L_x}^{L_x}\right\| < e) \end{aligned} \tag{12}$$

where $\Pr(\cdot)$ denotes probability and $\|\cdot\|$ denotes the maximum norm.

The strict Granger noncausality condition in Eq. (12) can be expressed as

$$\frac{C1(m + L_x, L_y, e)}{C2(L_x, L_y, e)} = \frac{C3(m + L_x, e)}{C4(L_x, e)}, \tag{13}$$

where joint probabilities can be represented as

$$\begin{aligned} C1(m + L_x, L_y, e) &\equiv \Pr(\| X_{t-L_x}^{m+L_x} - X_{s-L_x}^{m+L_x} \|< e, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \|< e), \\ C2(L_x, L_y, e) &\equiv \Pr(\| X_{t-L_x}^{L_x} - X_{s-L_x}^{L_x} \|< e, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \|< e), \\ C3(m + L_x, e) &\equiv \Pr(\| X_{t-L_x}^{m+L_x} - X_{s-L_x}^{m+L_x} \|< e), C4(L_x, e) \equiv \Pr(\| X_{t-L_x}^{L_x} - X_{s-L_x}^{L_x} \|< e). \end{aligned} \tag{14}$$

By using the correlation-integral estimators to estimate the joint probabilities mentioned above, the null hypothesis for $\{Y_t\}$ strictly Granger-causing $\{X_t\}$ in Eq. (13) is

$$\sqrt{n}(\frac{C1(m + L_x, L_y, e, n)}{C2(L_x, L_y, e, n)} - \frac{C3(m + L_x, e, n)}{C4(L_x, e, n)}) \overset{a}{\sim} N(0, \sigma^2(m, L_x, L_y, e)). \tag{15}$$

Table 2: Nonlinear Granger Causality Test
The nonlinear Granger causality test is applied to the VAR residuals corresponding to the return series and the volume series. The lead length is set at $m = 1$ and $e = 1.5\sigma$, $\sigma = 1$.
The test statistics CS and TVAL here respectively denote the difference between the two conditional probabilities in Eq. (13) and the standardized test statistic in Eq. (15).

$L_x = L_y$	H_0: ΔV Do Not Cause Δp			H_0: Δp Do Not Cause ΔV		
	CS	TVAL	p-value	CS	TVAL	p-value
1	0.2212	19.4902	0.0000	0.0387	3.3011	0.0004
2	0.2373	13.2561	0.0000	0.0784	3.8632	0.0000
3	0.2304	8.0993	0.0000	0.0537	1.4892	0.0682
4	0.2441	4.4353	0.0000	0.0970	1.3924	0.0817
5	0.3561	3.6677	0.0001	0.1168	0.9407	0.0734
6	0.4578	2.8371	0.0023	0.0833	1.4286	0.0334
7	0.6250	2.9589	0.0015	0.0875	2.8310	0.0023
8	0.6667	2.4520	0.0071	0.2857	1.5664	0.0944

7.2 *Test on simulation results*

To test the nonlinear Granger causality between the price and the trading volume, linear VAR models are firstly applied on both the simulated price series and the corresponding volume series to remove any linear predictive power and obtain two estimated residual series $\{U_{p,t}\}$ and $\{U_{V,t}\}$. Secondly, the results from an augmented Dickey–Fuller test indicate the existence of autocorrelation in both series, which suggests that we should take the differences until we generate two stationary series. Following Hiemstra and Jones (1994), the lead length is set at $m = 1$ and $e = 1.5\sigma$, $\sigma = 1$. We used the time series from $t = 0$ to $t = 500$, $p_0 = 61.66$. The test results with different sets of L_x and L_y are presented in Table 2.

Our results clearly demonstrate strong evidence of unidirectional nonlinear Granger causality from trading volume to stock returns under 1% level of significance. However, the results from stock prices to trading volume are not as significant. It is only safe to conclude the existence of nonlinear Granger causality in all circumstances under the level of significance of 10%.

8 Conclusion

In this paper, a simple model with agents holding heterogeneous beliefs is constructed to replicate most of the characteristics of price–volume movements widely known by speculators. Under the framework of market makers, the model regards both price and volume as functions of excess demand (or supply) from fundamentalists and chartists. It shows, without imposing any external assumptions such as price distributions or risk preferences, that the interaction between heterogeneous agents is sufficient on its own to reflect most of the observed price–volume relations.

Our simulation results are comprehensive on three different levels. Firstly, the generalized heterogeneous agent model is able to replicate satisfactorily the price–volume relations, both the trading volume with the absolute change of price and the trading volume with the price change itself. Secondly, since volume is highly valued by technicians, to provide the rationale behind the charting, price–volume signals in chart patterns are also successfully simulated. Thirdly, this model provides a persuasive explanation for volume signals on price changes by introducing the sudden decline zones, disturbing decline zones, and smooth decline zones.

Further exploration could also be made. In the strategy of each group, none of the investors takes into consideration the trading volume and tests the self-fulfilling power of the volume signals. Moreover, as regards the decision-making process, whether the fundamentalists are able to beat the chartists at making a profit is always an interesting topic. According to the EMH (Efficient Market Hypothesis), the chartists should be the ultimate losers and exit the market. However, is it necessarily so? Our simple heterogeneous agent model may be useful in answering this question.

Appendix: chance function in $\alpha(p)$

In section 3, in the definition of the excess demand of fundamentalists, the chance function $A(u_t^f, p_t)$ is introduced, which was first set out by Day and Huang (1990). It is a bimodal probability density function of u_t^f and p_t with peaks near the extreme values m and M, and aims to illustrate our setting of the psychological behaviors of the fundamentalist. It represents the chance of a lost opportunity where, when the current price p_t is closer to the topping price M, the fundamentalist believes that the probability of losing a capital gain and experiencing a capital loss will be higher. On the other hand, if p_t is close to the bottoming price m, the probability of missing a capital gain by failing to buy is also higher. Therefore, the chance function can be depicted as

$$A(p_t) = a(p-m)^{d_1}(M-p)^{d_2}. \tag{16}$$

Here, a is the magnitude that describes how sensitive the fundamentalists will be when the price nears their psychological boundaries. The extreme values m and M in Eq. (16) are the same boundaries of the price fluctuations as in Eq. (1). A simple illustration of the chance function can be observed in Fig. 8. Further details on the chance function can be found in Day and Huang (1990).

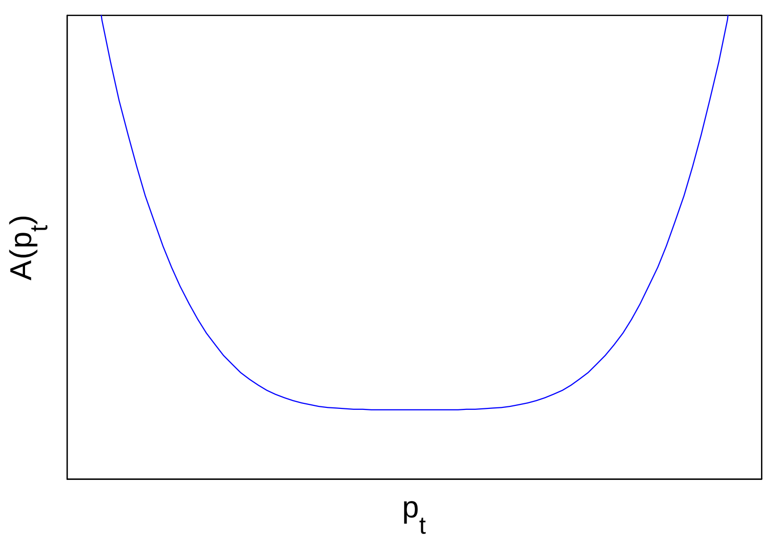

Fig. 8: Simple illustration of the chance function when the u_c^f is a constant.

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