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Relative Profitability of Dynamic Walrasian Strategies

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The advantage of price-taking behavior in achieving relative profitability in oligopolistic quantity competition has been much appreciated recently from economic dynamics and evolutionary game theory, respectively. The current research intends to provide a direct economic interpretation as well as intuitive justification and further to build a linkage between different perspectives. In particular, a detailed illustration of an arbitrary oligopoly that produce a homogenous product is presented. So long as the outputs of other firms are fixed and the residual demand is downward sloping, for any two identical firms whose cost functions are convex, their output space can be divided symmetrically into mutually exclusive relatively profitability regimes. Furthermore, there exist infinitely many relative-profitability reactions for each firm in such "residual" duopoly, all of which intersect at the "residual" Walrasian equilibrium. This suggests that sticking to this dynamical equilibrium output constantly (i.e., the static Walrasian strategy) turns out to be a relative-profitability strategy at each period. On the other hand, regardless of what strategies its rival may take, a firm adopting price-taking strategy or more generally defined dynamic Walrasian strategies can achieve the relative profitability if an intertemporal equilibrium is reached. The methodology adopted and the conclusions arrived clarify the confusions and misunderstandings due to the different usages of same terminologies under different frameworks and generalize the previous available results in the literature to a higher level and a broader context.

Key Words: Price-taking, Walrasian behavior, Relative profit, Oligopoly, Cournot, dynamic Walrasian strategy

Contents

1. INTRODUCTION

One of the lessons learned from basic microeconomics is that a profit maximizing firm will always leverage on market information as well as the behavioral rule of its rivals when making its output decision. Given limited information about the market as well as its rivals' behavior (which varies from case to case), it is invariably assumed that a quantity competing oligopolistic firm's best response is to maximize myopically its instantaneous absolute payoff, giving rise to a reaction function of its rival's expected output for the period. It is deemed economically irrational if a firm either ignores or is ignorant about its market power and instead adopts some simple strategy in oligopolistic competition.

However, such beliefs were questioned and challenged from different perspectives over last decade. A huge body of literature from evolutionary game theory revealed that producing at the competitive equilibrium output level (or adopting static Walrasian strategy) is an evolutionarily stable strategy (ESS) when all firms have identical technology. The study of Cournot oligopolies where firms learn through imitation of success can be traced back as early as to Alchian [1950]. Schaffer (1989) demonstrates with a Darwinian model of economic natural selection and shows that the profit-maximizers are not necessarily the best survivors. In a simple context with just two quantity-setting firms which have identical and constant marginal costs, only price-taking behavior is evolutionarily stable. Vega-Redondo (1997) shows that, for a symmetric oligopoly in which all firms have identical non-decreasing return technology, when some firms produce at the competitive equilibrium level and the others produce at an same identical mutant level (the mutant strategy), the former will gain relative higher profit than the latter. This result is independent of both the number of mutant firms and of the mutant outputs so long as they are identical. The ESS characteristics of static Walrasian strategy have lately been extended in different directions under different specifications by Schenk-Hoppe (2000) and Alos-Ferrer and Ania (2005), etc.

The relative profitability of price-taking behavior in the dynamic sense where a firm produces at an output level that equates the current marginal cost to price-expectation, has been explored in Huang (2002), where it is showed that, in an oligopolistic industry where all firms equipped with identical technology[1], the price-takers always triumph over the other firms at a dynamical equilibrium in terms of relative profitability, regardless of what strategies (dynamical behavioral rules) these non-price-takers may adopt. Therefore, the price-taking behavior is appreciated without the critical assumption of "imitations" in evolutionary game theory analysis.

The aims of current research are i) to study the relative profitability of Walrasian strategies in general under an unified framework so that intuitive economic interpretations can be justified for both static and dynamic Walrasian strategies and mutual implication are explored. ii) to generalize the previous results to more general oligopolistic industry with heterogeneous technologies.

All these goals are achieved through characterizing the relative-profitability regime in the output space. In an arbitrary oligopoly, so long as the outputs of other firms are fixed, for any two firms who share an identical technology

[1] It will be made clear later that the assumption of uniform technology for all firms are not necessary.

that exhibits non-increasing return to scale, their output space can be divided uniquely into symmetrically located as well as mutually exclusive relative-profitability regimes. For each firm in such "residual" duopoly, there exist infinitely many relative-profitable strategies, among which are the dynamic Walrasian strategies and a unique static Walrasian strategy. While adopting static Walrasian strategy can achieve relative profitability against any other strategies at each period of dynamical adjustment so long as the market environment is fixed, the dynamic Walrasian strategies, which demand minimum market information, possess relative-profitability only when an intertemporal equilibrium is reached. Nevertheless, the dynamic Walrasian strategies are robust to changes in market environment such as residual demand, the entry and exit of oligopolistic firms, the technological advances et. al. Finally, the dynamic Walrasian strategy would converge to the static Walrasian strategy, should firms imitate each other and strive for the relative successes as assumed in the evolutionary game theory.

The remaining discussion is organized as follows. Section 2 defines the concept of relative-profitability curve and relative-profitability regimes in a symmetric residual-duopoly. The uniqueness of relative-profitability curve is ensured providing that the cost function is convex. The concepts of relative-profitability strategy and efficient relative-profitability, i.e., the strategy that brings about the maximum profit difference, are introduced in Section 3, in which the compromise of relative-profitability with absolute profitability is explored. Section 4 is the core of this article, in which the concept of dynamic Walrasian strategy is formally defined and its linkage to the static Walrasian strategy discussed in evolutionary game theory is built. Concluding remarks as well as the further research are provided in Section 6. For the sake of presentation, some straightforward or tedious analytical derivations are included in Appendix A. Typical characteristics of relative-profitability curve and Walrasian reaction curve are provided in Appendix B and C, respectively.

2. RELATIVE-PROFITABILITY FRONTIER

Consider an oligopoly market, in which N firms produce a homogeneous product with quantity q_t^i, $i = 1, 2, ..., N$, at period t. The market inverse demand for the product is given by $p_t = \tilde{D}(\tilde{Q}_t)$, where $\tilde{D}' \leq 0$. The conventional assumption that $\tilde{Q}_t = \sum_{i=1}^{N} q_t^i$, i.e., *the actual market price adjusts to the demand so as to clear the market at every period* applies.

Our goal is compare the relative profitability of any two firms that have an identical technology exhibiting non-increasing returns to scale and thus an identical convex cost function $C(q)$, with $C'' \geq 0$, in such an oligopoly. To concentrate on the interaction between these two firms, we refer them as X and Y, and assume that the output levels of all other firms are fixed so the *residual market demand* for these two firms is $D = D(x+y) \doteq \tilde{D}(x + y + \tilde{Q}')$, where x and y denote the outputs of these two firms and let $\tilde{Q}' \doteq \sum_{j \backslash x,y} q^j = \tilde{Q} - x - y$ denote the outputs of all other firms. For the convenience of latter reference, we shall call X and Y as a residual-duopoly and similarly use prefix "residual-" on relevant terminologies to indicate that the analysis and the conclusion arrived for any two identical firms in an oligopoly with all other firm's outputs being fixed.

We start with some basic definitions for the residual-duopoly.

2.1. Relative-profitability frontier

Let $\mathbb{Q}$ be the economically meaningful domain for x and y when $\tilde{Q}'$ is given. With $D' < 0$ implied by $\tilde{D}' < 0$, the profits gained by firms X and Y, are then given by:

$$\pi^q(x, y) = D(x + y)\, q - C(q)\,, \text{ with } q = x, y \in \mathbb{Q}.$$

Denote the *relative profit* for Firm X, $\Delta^{xy}(x, y)$, as its profit difference as compared to Firm Y, that is,

$$\Delta^{xy}(x, y) \doteq \pi^x(x, y) - \pi^y(x, y)\,.$$

DEFINITION 1. *Equal-profitability curve* in x-y plane refers to the curves in the *economically meaningful domain* which give rise to $\pi^x = \pi^y$, that is, $\Delta^{xy}(x, y) = \Delta^{yx}(x, y) = 0$.

DEFINITION 2. *Relative profitability regime* (for Firm X) is a (x, y) subset in the *economically meaningful domain* $\mathbb{Q}$ in which Firm X makes a higher profit relative to his rival Firm Y, that is, $\Delta^{xy}(x, y) > 0$.

The 45-degree diagonal line in x-y plane or equivalently, $y = x$, is a trivial equal-profitability curve (which arises from the fact that both firms have an identical cost.) In addition to this trivial equal-profitability curve, there also exist non-trivial equal-profitability curves.In Figure 1, with residual demand being fixed at $D(Q) = 1/Q$, typical 3-dimensional plots for Δ^{xy} (positive portion only) are provided for three different cost functions: (a) strict convex cost: $c_1(q) = cq^2/2, c > 0$; (b) strict concave cost: $c_2(q) = cq, c > 0$ and (c) S-shape cost (convex-concave) cost: $c_3(q) = c\sin^2(\pi q/d)\,, c, d > 0$.

For a general residual-duopoly, the first question that interests us is whether this non-trivial equal-profitability curve is unique when the residual demand function and cost function satisfy certain requirements. A definite and positive answer is provided in the next theorem. To distinguish this unique non-trivial equal-profitability curve with the trivial one, we shall call it relative-profitability frontier hereafter.

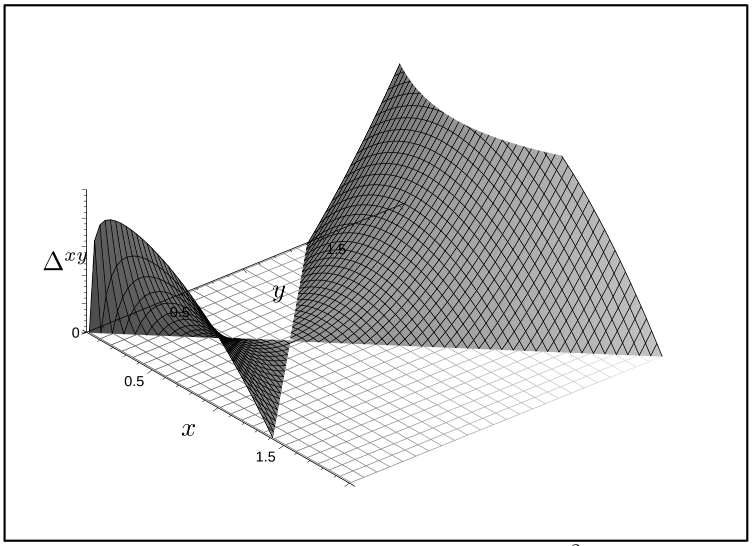
DEFINITION 3. **Relative-profitability frontier**: all (x, y) combination in the *economically meaningful domain* $\mathbb{Q}$ such that $\Delta^{xy}(x, y) = 0$ but $x \neq y$.

A critical concept that relates to the relative-profitability frontier and plays an important role in the proof of the following theorem is the Walrasian reaction curve.

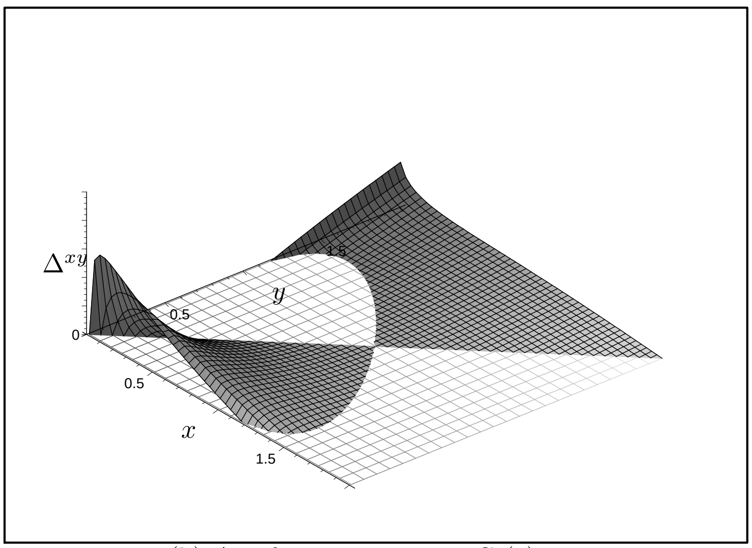
DEFINITION 4. **Walrasian reaction curve** By Walrasian reaction for Firm X, we mean that for any given rival's output y, Firm X responds by equating its marginal cost to the expected price. That is, the reaction to a given y, denoted as R_w, is implicitly defined by

$$D(R_w(y) + y) = C'(R_w(y))\,. \tag{1}$$

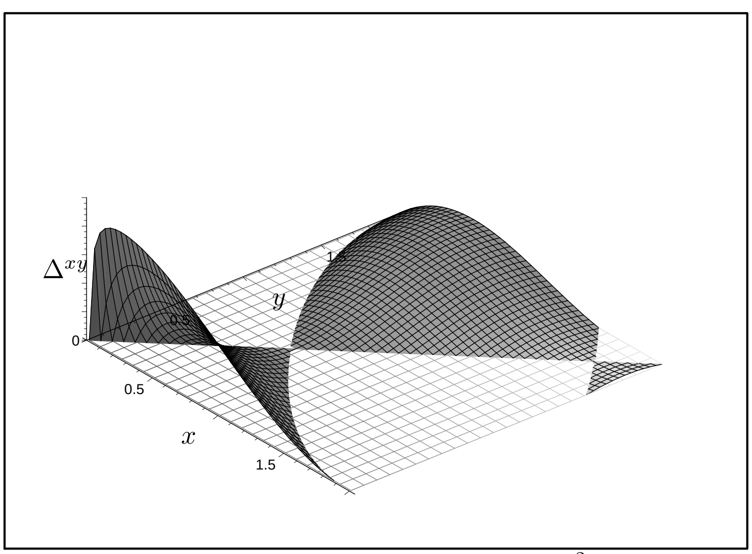
THEOREM 1. ***(Uniqueness Theorem)*** *When C is convex, that is, $C''(q) \geq 0$ for all $q \in \mathbb{Q}$, and $D' < 0$, we have the following*



(a) Δ^{xy} for convex cost: $C_1(q) = q^2/2$



(b) Δ^{xy} for concave cost: $C_2(q) = \sqrt{q}$



(c) Δ^{xy} for S-shape cost: $C_3(q) = sin^2(\frac{\pi}{4}q)$

FIG. 1 Profit Difference $\Delta^{xy}(x, y)$ $(D(Q) = 1/Q)$

i) there exists a unique relative-profitability frontier given by f_e such that $\Delta^{xy}(x, f_e(x)) = 0$ and $x = f_e(x)$ if and only if $x = q_w$, where q_w is uniquely determined from the following identity

$$C'(q_w) = D(2q_w).$$

ii) $f_e'(x) \leq 0$, where the equality holds only at finite points;

iii)with the trivial equal-profitability curve and the relative-profitability frontier f_e, the economically meaningful domain $\mathbb{Q}$ in x-y plane is divided into four quadrants, the vertex of which is characterized by the residual Walrasian equilibrium $E_w = (q_w, q_w)$. While the upper and lower quadrants forms the relative-profitability regime for Firm X, the left and the right quadrants is the relative-profitability regime for Firm Y.

Proof. (see Appendix A) ∎

Figure 2 depicts several examples of the relative-profitability frontiers (as well as the Walrasian reaction curves). It needs to mention that, for a duopoly, the residual Walrasian equilibrium simplifies to the competitive equilibrium.

Remark 1. For monotonically downward sloping demand $D' < 0$, the convexity of cost, that is, $C'' \geq 0$, is a sufficient condition for the uniqueness (as well as the monotonicity) of relative-profitability frontier f_e. If the convexity of cost can't be warranted, there exist situations where multiple non-trivial equal-profitability curves coexist, as illustrated in Figure 1.(c), as well as the possibility that the non-trivial equal-profitability curve is not monotonic, as illustrated in Figure 1.(b).

2.2. Characterize the relative-profitability frontier f_e

Due to the fact that $\pi^x(x, y) \equiv \pi^y(y, x)$, the relative-profitability frontier f_e must be symmetric with respect to the 45-degree line. This kind of anti-symmetricity is analytically characterized by

$$f_e^{-1}(x) = f_e(x)$$

which demands that $f_e'(q_w) = -1$, where q_w is the residual Walrasian output. Formally, we have

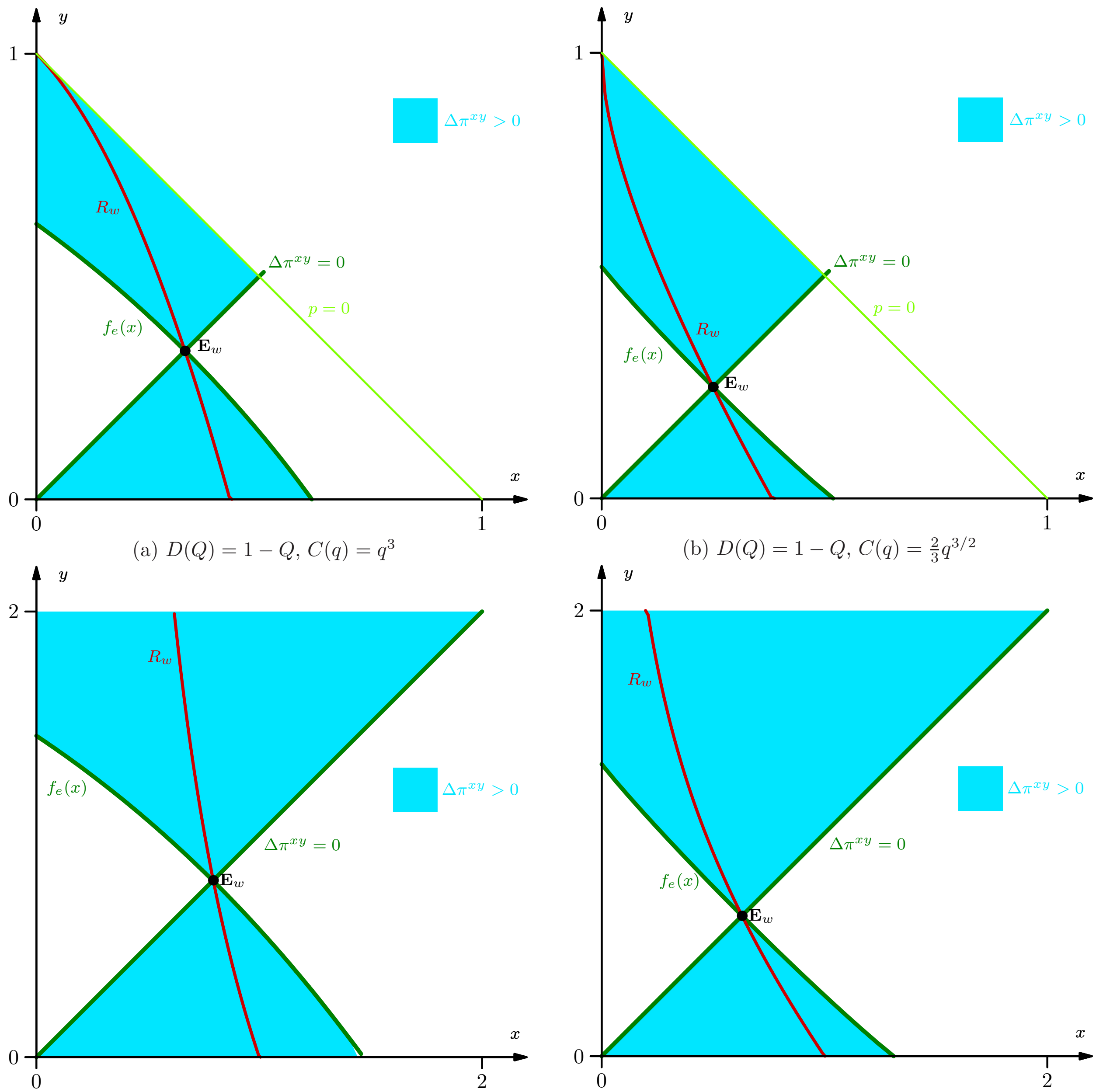
PROPOSITION 1. *The relative-profitability frontier f_e is symmetric with respect to the 45-degree line, whose derivative is given by*

$$f_e'(x) = \frac{C'(x) - D(x+y) - (x-y) D'(x+y)}{C'(y) - D(x+y) + (x-y) D'(x+y)} < 0 \tag{2}$$

and $f_e'(q_w) = -1$.

The second-order derivative of f_e has very complicated analytical expression, which makes the analysis of the second-order derivative property of f_e impossible. We shall bypass this obstacle by evaluating the relative magnitude of absolute value of f_e' in comparison to unity. Due to the symmetricity of f_e, we only need to discuss the lower portion of f_e, that is $x \geq q_w \geq y$. It follows from

$$|f_e'| - 1 = \frac{C'(x) + C'(y) - 2D}{(D - C'(y)) - (x-y) D'},$$



(a) $D(Q) = 1 - Q$, $C(q) = q^3$

(b) $D(Q) = 1 - Q$, $C(q) = \frac{2}{3}q^{3/2}$

(c) $D(Q) = 1/Q$, $C(q) = q^3$

(d) $D(Q) = 1/Q$, $C(q) = \frac{2}{3}q^{3/2}$

FIG. 2 Illustrative relative profitability regimes as well as the fact that $D(x+y) = (C(x) - C(y))/(x-y)$ when $y = f_e(x)$, we have

PROPOSITION 2. *For convex C, we have $|f_e'| \lesseqgtr 1$ for all $x \geq q_w \geq y$ if the inequality*

$$\frac{C'(x) + C'(y)}{2} \lesseqgtr \frac{C(x) - C(y)}{x - y} \tag{3}$$

holds for all $x > y > 0$.

Condition (3) provides us with the information about the concavity of f_e near the residual Walrasian equilibrium due to the fact that $\lim_{x \to q_w} f_e'' \gtreqless 0$ if and only if $\lim_{x \to q_w^-} |f_e'| \lesseqgtr 1$. Geometrically, the non-trivial equal profitability curve f_e is convex (concave) around E_w if the average marginal costs for any two points exceeds the slope of chord that connects the corresponding points in the cost function C.

For instance, if $C(q) = cq^\alpha$, $\alpha \geq 1$, let $F(x,y) = (C'(x) + C'(y))/2 - (C(x) - C(y))/(x-y)$, then we have

α	$F(x,y)$	f_e''
1	0	0
$3/2$	$-\frac{1}{4}c\frac{\sqrt{x}(x+3y) - \sqrt{y}(y+3x)}{x-y} < 0$	$+$
2	0	0
3	$c\frac{1}{2}(x-y)^2 > 0$	$-$

We see that $f_e'' = 0$ if $\alpha = 1, 2$. For $\alpha = 3/2$ and 3, the relative-profitability frontiers are convex and concave, respectively, as shown in Fig. 2 for two different residual demand functions.

The quadratic cost is most widely adopted as example in the economical analysis. The proof of the following interesting fact is provided in Appendix A.

PROPOSITION 3. *When cost function C is quadratic, that is, $C(q) = a + bq + cq^2$ with $a, b, c \geq 0$, then the relative-profitability frontier f_e is always a straight line with slope -1 so long as the residual demand D satisfies the basic requirement $D(0) > b$. Moreover, when the cost is linear ($c = 0$), the relative-profitability frontier geometrically coincides with the Walrasian reaction R_w.*

The relative-profitability regimes expand and shrink with changes in the residual demand as well as the cost function. Understanding the impacts of the various changes in market characteristics on the shape and location of f_e enable us to study the comparative statics when technology changes as well as when the entry and exit are allowed. The following proposition can be easily verified.

PROPOSITION 4. *Comparative effects of the relative-profitability frontier*

i) For the same residual demand function D, let $f_e^{(i)}$ be the relative-profitability frontier when $C^{(i)}$ is given as the cost function of the residual duopoly, $i = 1, 2$. If $C^{(2)}(q) \geq C^{(1)}(q)$ for all $q \in \mathbb{Q}$, then we have $f_e^{(2)}(x) \leq f_e^{(1)}(x)$ for all $x \in \mathbb{Q}$. In other words, increasing the cost of residual-duopoly shifts the relative-profitability frontier downward.

ii) For a given identical cost C, let $f_e^{(i)}$ be the relative-profitability frontier when the residual demand is given by $D^{(i)}$, $i = 1, 2$. If $D^{(2)}(x+y) \leq D^{(1)}(x+y)$ for all $x, y \in \mathbb{Q}$, then we have $f_e^{(2)}(x) \leq f_e^{(1)}(x)$ for all $x \in \mathbb{Q}$. In other words, increasing the residual demand shifts the equal-profitability curve upward.

3. RELATIVE-PROFITABILITY STRATEGIES

In the theory of firms, all firms are commonly assumed to seek maximized (absolute) profit. However, economists have pondered over the definition and/or criteria of profit for quite some time (Bernstein 1953). A directly related question to the definition and/or criteria of profit is apparently what are the firms' goals. This question, nevertheless, also remained very much an open debate in the history of economics (Osborne 1964). In Baumol's seminal work (Baumol (1959)), it was suggested that in the real world, a firm is actually maximizing the sales revenue subject to minimum profit requirement rather than maximizing absolute profit. The pioneering work on the theory of bounded rational behavior by Simon (1959) further stimulated the discussion of firm's objectives. The view that "the firm seeks to attain a satisfactory level of profits rather than a maximum level" had received overwhelming responses in 1960s. As Lamberton (1960) had argued, "*when dealing with the large, multi-product, oligopolistic firm whose particular (and general) expectations are held with uncertainty the hypothesis is clearly a plausible one. Business management will frequently be thinking in terms of simultaneous, discontinuous changes in a large number of variables with which it is concerned and will have recourse to conventional procedures, one which may be the adoption of a profit target....The target may be indicated according to a variety of methods, e.g., percentage on turnover or capital employed, and pursued by a single policy under stable conditions*."

The relevance of profit maximization is much less obvious for large modern corporations where ownership and control of the firm are separated: the former in the hands of potentially diffused shareholders and the latter vested in professional management. This separation provides a considerable degree of decision-making autonomy of managers, whose behavior may deviate significantly from what is implied by profit maximization. It is well known in the strategic-managerial-incentives literature that a firm's owner can increase the firm's profit by hiring a manager and assigning him an objective different from profit-maximization (Vickers (1985), Fershtman and Judd (1987) and Sklivas (1987)). A literature relevant to relative profitability is from Lundgren (1996), where a method to eliminate incentives for collusion by making managerial compensation, which depends on relative profits rather than absolute profits, is proposed. This alteration of managerial incentives sets up a zero-sum game among the firms in an industry, yielding the result that firms no longer have incentive to collude, either actually or tacitly, with regards to prices or outputs.

Recently, increasing appreciation of Walrasian equilibrium output from evolutionary and game theoretic perspectives have once again provided alternative views to profit maximization as firms' primary objectives. The essence of evolutionary economics lies in the evaluation of relative success of different strategies. Rational firms imitate the successful strategies, i.e., the strategies that can achieve a profit higher than the average. Thus, during an evolutionary process, it is not the absolute profit but the relative profit that matters. Therefore, pursuing a goal of achieving relative profit advantage is an economically justified in oligopolistic competition. A detailed examination of this concept is warranted.

3.1. Relative-profitability strategy

DEFINITION 5. *Relative-profitability reaction (response)* Given a rival's output level, the firm reacts with an *output* that can lead to a higher relative profit at each period during the dynamically oligopolistic competition.

Relative-profitability strategy In dynamic oligopolistic competition, a relative-profitability strategy is any open-loop or close-loop strategy such that all the equilibrium outputs lie on a relative-profitability reaction curve.

It needs to emphasize that the relative-profitability strategy is a broader concept than the relative-profitability reaction. With full information about the current output of its rival, a firm responding with the relative-profitability reaction is adopting a relative-profitability strategy. A relative-profitability strategy, however, can be composed of finite relative-profitability responses (to current or past outputs of its rivals), or a simple response to the past price-level, or even producing at a constant output like q_w.

Geometrically, all relative-profitability reactions for Firm X must lie in the relative-profitability regime for Firm X. For each firm of residual-duopoly, there exist infinitely many relative-profitability strategies, all of them must intersect at the residual Walrasian equilibrium. Then the natural question that arises is which one of them can maximize the relative profit, should a firm's goal be to *maximize* the relative profit intentionally. This section is conducted under such motivation.

DEFINITION 6. *Efficient relative-profitability reaction* is the relative-profitability reaction that maximizes the relative profit for any given rival's output.

Efficient relative-profitability strategy is any dynamical strategy that guarantees that a dynamical equilibrium will lies on the efficient relative-profitability reaction curve.

The efficient relative-profitability reaction therefore reflects the maximum possible difference in profit when the rival's output is given. This reaction naturally requires information about the residual demand function. With knowledge of the residual demand D and the rival's output y, Firm X maximizes its relative profit given by $\Delta^{xy} = (x-y)\, D\,(x+y) - (C\,(x) - C\,(y))$. A local maximum of Δ^{xy} occurs at

$$C'\,(x) = D + (x-y)\, D' \tag{4}$$

if $\partial^2 \Delta^{xy} / \partial x^2 = 2D' + (x-y)\, D'' - C''\,(x) < 0$.

Denote the efficient relative-profitability reaction implicitly defined in (4) as $x = R_r\,(y)$, and the Cournot best-response as $x = R_c\,(y)$, which is implicitly defined by the first-order profit maximization condition as

$$C'\,(x) = D + xD'. \tag{5}$$

We should be able to infer from (4) and (5) that $R_r(y) > R_c(y)$ for all $y > 0$.

However, in comparing with the Walrasian reaction $R_w(y)$ defined in (1), we have the following generic observation:

THEOREM 2. *If $C'' \geq 0$, we have $R_w(y) > R_r(y) > R_c(y)$ for $y < q_w$ and $R_r(y) > R_w(y) > R_c(y)$ for $y > q_w$.*

Proof. See Appendix A. ∎

If $C''(q) \equiv 0$ for all $q \in \mathbb{Q}$, that is, $C'(q) \equiv c$, $R_w(y)$ coincides with the equal-profitability curve as well as the normal-profitability curve. Therefore, for any $y > q_w$, with efficient relative-profitability reaction, both firm X and Y make negative profit while Firm Y loses more. To maintain a positive profit, the output of Firm Y must be less than q_w.

Remark 2. The efficient relative-profitability strategy $x = R_r(y)$ implicitly defined in (4) is independent of the cost function of Y and hence can be generalized to the cases in which two firms have different cost functions.

EXAMPLE 1. For a residual-duopoly consisting of Firm X and Y, assume the cost function is given by $C(q) = cq^2/2$. When the residual demand take the two forms that are most commonly seen: linear demand $D(Q) = 1 - Q$ and iso-elastic demand $D(Q) = 1/Q$, the relevant reaction curves are summarized in the following table.

	$D(Q) = 1 - Q$	$D(Q) = 1/Q$
$f_e(x)$	$\dfrac{2 - x(2+c)}{2+c}$	$\sqrt{2/c} - x$
$R_c(y)$	$(1-y)/(2+c)$	$\dfrac{u_c(y)}{6c} + \dfrac{2}{3}\dfrac{y^2 c}{u_c(y)} - \dfrac{2}{3}y$
$R_w(y)$	$(1-y)/(1+c)$	$\left(\sqrt{y^2 + 4/c} - y\right)/2$
$R_r(y)$	$1/(2+c)$	$\dfrac{u_r(y)}{3c} + \dfrac{1}{3}\dfrac{cy^2}{u_r(y)} - \dfrac{2}{3}y$
q_w	$1/(2+c)$	$\sqrt{2/c}$
Note:	$u_r(y) \doteq \sqrt[3]{y\left(cy^2 + 27 + 3\sqrt{3(2cy^2+27)}\right)c^2}$ $u_c(y) \doteq \sqrt[3]{4y\left(2cy^2 + 27 + 3\sqrt{3(4cy^2+27)}\right)c^2}$	

Table 1: Reaction curves with $C(q) = q^2/2$

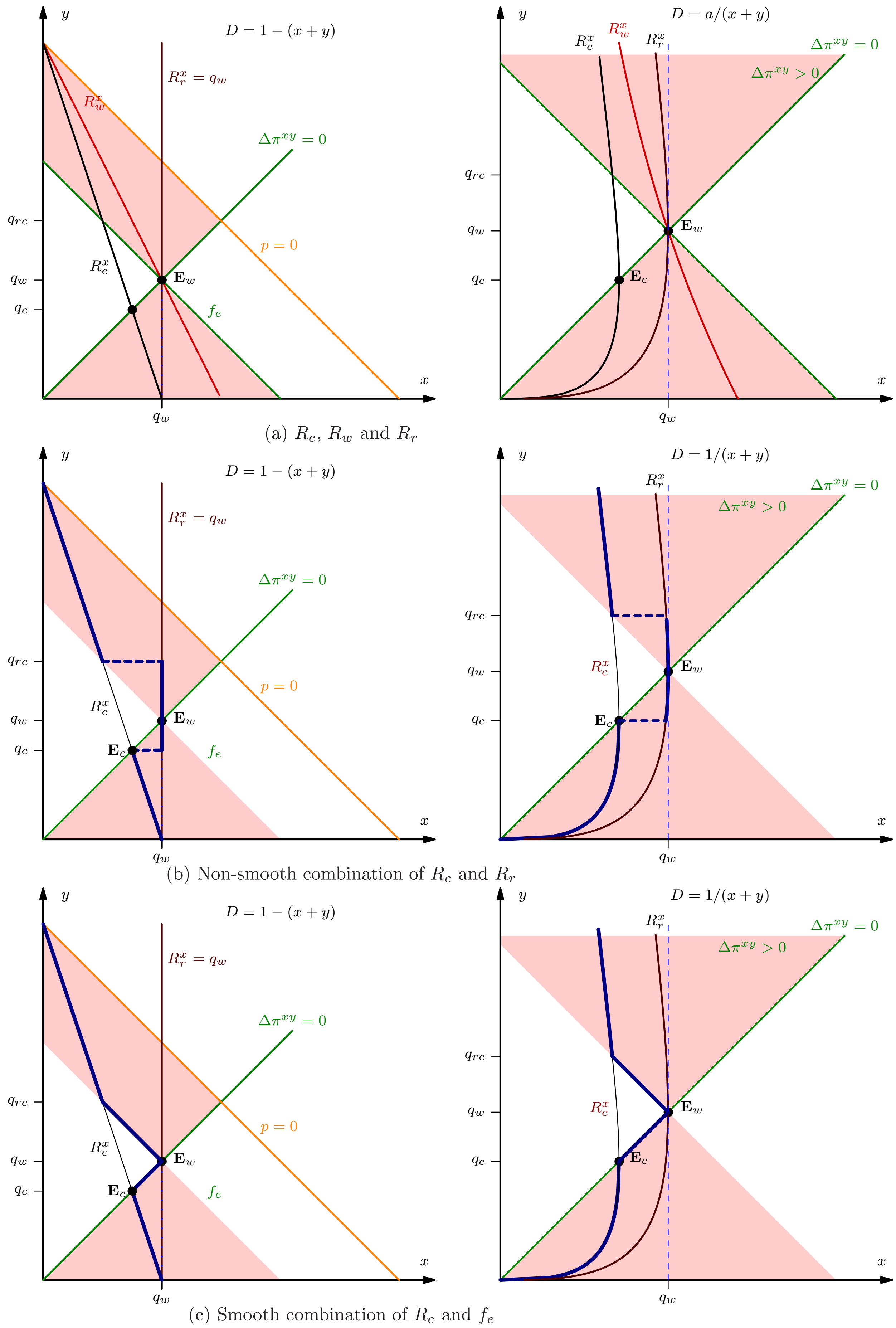
The relevant curves are illustrated in Figure 3 (a), from which we make the following observations for both cases:

i) $R_w(y) > R_r(y) > R_c(y)$ for $y < q_w$ and $R_r(y) > R_w(y) > R_c(y)$ for $y > q_w$, as suggested in Theorem 2;

ii) Relative-profitability frontier f_e is a straight line, as implied by Proposition 3

iii) When the residual demand is linear, $R_r = q_w$, a fact to be verified in Proposition 4 of next section.

iv) $\Delta^{xy}(q_w, y) > 0$, if $y \neq q_w$. For any given y, $\Delta^{xy}(x, y)$ increases from zero (when $x = y$) to its maximum value at $x = R_r(y)$ and then decreases to zero at $x = f_e$ after x passes the Walrasian reaction f_w.



(a) R_c, R_w and R_r

(b) Non-smooth combination of R_c and R_r

(c) Smooth combination of R_c and f_e

FIG. 3 Relative profitability and Absolute profitability

3.2. Reconciling relative-profitability with absolute profitability

As have been discussed, a firm's objective may be to maximize if not simply maintain the relative-profitability. Condition on the fact that a firm's main priority is to maintain relative-profitability, we ask if the firm is able to maximize absolute profit at the same time. The answer is positive when the rival's output falls in the certain range.

Actually, it can be verified that if a firm adopts Cournot best-response in a residual duopoly, two sections of this best-response actually pass through its relative-profitability regimes, one is below and the other is above the residual Walrasian equilibrium. In other words, any point on these two sections of the best response function not only allows the firm to achieve the highest absolute profit given its rival's output level but also ensures that the firm makes higher profit relative to its rival. Therefore, these two sections of best-response themselves can in fact be interpreted as *the relative-profitability strategy that achieves the highest absolute profit level.* When the main objective of relative profitability is not met, the firm is willing to give up the opportunity of making the maximum absolute profit and thus responds differently from Cournot-best response. Such compromise can be supplemented either with an equal profitability (so that to main certain absolutely profitability) or with efficient relative-profitability response, depending on the profitability preference of the firm.

Example 2. Following the settings of Example 1.

A direct combination of R_c and R_r can be constructed with

$$x = R_{cr}(y) = \begin{cases} R_c(y), & y \in [0, q_c) \cup [y_{rc}, \infty), \\ R_r(y), & y \in [q_c, y_{rc}], \end{cases}$$

where y_{rc} corresponds to the y value at the intersection point of R_c and f_e. When $y \in [q_c, y_{rc}]$, the firm cannot achieve relative profitability if it adopts the Cournot-best response and thus it switches to maximizing the relative profit. Its reaction function is thus $R_r(y)$. Otherwise, since the goal of relative profitability is achieved, it will then sought to maximize the absolute profit by selecting the Cournot-best response, which is given by $R_c(y)$. We notice that the discontinuity inevitably occur at $y = q_c$ as well as at $y = y_{rc}$, as illustrated by the thick curve in Figure 3.(b)

Alternatively, the firm may choose to maximize absolute profit given that relative profitability is maintained when $y \in [q_c, y_{rc}]$.To achieve this secondary objective, the firm will choose to react according to y when $y \in (q_c, q_w)$ and $f_e^{-1}(y)$ when $y \in (q_w, y_{rc})$, where $f_e^{-1}(y)$ is the inverse of f_e. The reaction is thus summarized by

$$x = R_{ce}(y) = \begin{cases} R_c(y), & y \in [0, q_c) \cup [y_{rc}, \infty), \\ y, & y \in (q_c, q_w), \\ f_e^{-1}(y), & y \in (q_w, y_{rc}), \end{cases}$$

and is illustrated by the thick curve in Figure 3.(c).

4. WALRASIAN BEHAVIORS IN GENERAL

Always stick to your proved strategy!

—Another Chinese Philosophical Quotation

In the proof of Theorem 1, we have actually shown that, when $x > (<)\, y$, the relative-profitability frontier f_e lies on the left (right) hand side of the inverse of the Walrasian reaction curve f_w, respectively. This in turn suggests that the Walrasian reaction $x = R_w(y)$, for any given y, lies in the relative-profitability regime of X, or equivalently, Walrasian reaction is one of the relative-profitability reactions. On the other hand, since the center of relative-profitability regime is the residual Walrasian equilibrium $E_w = (q_w, q_w)$, fixing the output level directly at the constant level of q_w is another simple relative-profitability strategy. This section will explore further the relative implications and extend our results to more general oligopolistic framework.

4.1. Dynamic Walrasian Strategies

In a residual duopoly, like relative-profitability reactions, there exist infinitely many relative-profitability strategies for each firm. Among them, the one that requires the minimum information about the market structure turns out to be the *price-taking strategy*, which is to react to only the lagged market price by adjusting current marginal cost:

$$p_{t-h} = C'(x_t),$$

or, equivalently,

$$x_t = MC^{-1}(p_{t-h}), \tag{6}$$

where p is the price and $h \geq 0$ is the information lag. The explanation for the delayed reaction function (6) being a relative-profitability strategy lies in the fact that when an equilibrium is reached with $x_t = x_{t-1}$, Eq. (6) is nothing but the Walrasian reaction defined in (1). The price-taking strategy certainly belongs to more general category of the Walrasian strategy, by which we mean:

DEFINITION 7. **Dynamic Walrasian strategy:** any dynamical strategy (or behavioral rule) that can guarantees the equality of marginal cost to the market price at any intertemporal equilibrium.

Geometrically, dynamic Walrasian strategy refers to any dynamical behavioral rule that can guarantee that an intertemporal equilibrium lies on the Walrasian reaction function for a residual duopoly. The dynamic Walrasian strategy so defined differs from the static one adopted in the evolutionary game theory. It includes the behavioral rules and reactions such as *simple trial and error procedure*, *imitating behavior*, *advanced learning rules*, *optimal or non-optimal search behavior*, *adaptive adjustments*, or even *dynamic optimizations* so long as they can reach to Walrasian reaction curve in a dynamic equilibrium.

A typical form of dynamic Walrasian strategy is the conventional adaptive adjustment defined by

$$x_t = x_{t-1} + \alpha\left(MC^{-1}(p_{t-1}) - x_{t-1}\right). \tag{7}$$

if the marginal cost is not a constant, where $\alpha \in (0,1)$ is the adaptive speed[2].

[2] An alternative, but more general adaptive adjustment strategy is

$$x_t = \alpha x_{t-1} + (1-\alpha)\, g\left(p_{t-1} - C'(x_{t-1})\right)$$

where g is a monotonically increasing function with $g(0) = 0$.

It needs to emphasize that the relative profitability of dynamic Walrasian strategy can be generalized to more general heterogenous oligopolistic models where firms have different costs and strategies. We have the following beautiful result[3]:

THEOREM 3. *In an oligopolistic economy consisting of N firms that produce an homogeneous product, assume Firm X has strict convex cost function C and adopts a dynamic Walrasian strategy. If an intertemporal equilibrium is arrived, Firm X can profit more than any rival who has identical cost but produces at different equilibrium output level.*

Proof. Assume Firm Y to be one of the firms having the identical cost C as the price-taker. Let $\bar{x}$ and $\bar{y}$ be the equilibrium outputs of the price-taker and Firm Y, respectively. Denote $\{\bar{q}_j\}_{j=3}^{N}$ as the equilibrium outputs for the other oligopolistic firms, *with either same or different costs*, so that the equilibrium price is given by

$$\bar{p} = \tilde{D}(\bar{x} + \bar{y} + \sum_{j=3}^{N} \bar{q}_j).$$

It follows from the definition of dynamic Walrasian strategy that $C'(\bar{x}) = \bar{p}$.

The profit difference between Firm X and Firm Y is then given by

$$\Delta^{xy}(\bar{x}, \bar{y}) = \bar{p} \cdot (\bar{x} - \bar{y}) - (C(\bar{x}) - C(\bar{y})) = C'(\bar{x})(\bar{x} - \bar{y}) - (C(\bar{x}) - C(\bar{y})).$$

It follows from the assumption of $C''(\cdot) > 0$ that $C'(\bar{x})(\bar{x} - \bar{y}) - (C(\bar{x}) - C(\bar{y})) \geq 0$, regardless the relative magnitudes of $\bar{x}$ and $\bar{y}$, or equivalently,

$$\Delta^{xy}(\bar{x}, \bar{y}) \geq 0, \tag{8}$$

where the equality holds if and only if $\bar{x} = \bar{y}$.

Since Firm Y's production strategy as well as the equilibrium output $\bar{y}$ are not explicitly specified, the inequality (8) thus leads to the conclusion immediately. ***Q.E.D.*** ∎

The conclusions drawn in Theorem 3 is generic in the sense that it depends on neither the particular market structure (the residual demand and cost functions) nor on the strategies (or outputs) that other firms may take. Moreover, since only equilibrium is concerned, the result is robust to the changes in the market environments such as the market demand, entry and exit of oligopolistic firms, advances in some or overall technology level. Hence, it supersedes all available conclusions in the relevant literatures. Economical interpretation for the relative profitability of dynamic Walrasian strategies in general is simple and straightforward: when an intertemporal equilibrium is reached, the price converges to an equilibrium and remains unchanged. Any sophisticated strategy aiming to affect the market through

[3] The relative profitability of the price-taking strategy was first formally proposed in Huang (2002) where the proof was mistakenly omitted in the editorial process. The proof itself is quite straightforward. Although the simplified forms were reported in several later publications, we again provide a more general form here for the sake of completeness as well as the appreciation of role of convex cost functions.

market power ceases to function and becomes in vain. In contrast, $MC = P$ is in fact the unique way to gain the economic efficiency, a fundamental economic principal taught to all undergraduates[4]

It should be emphasized that the relative profitability is not purposely but unconsciously achieved by a firm who adopts a dynamic Walrasian strategy. If a firm rival intends to maximize its profit relative to a price-taker by adopting any relative-profitability strategy, both firms will end up with producing at the residual Walrasian equilibrium level.

When a firm's production technology exhibits constant returns to scale, or, equivalently, the firm's marginal cost is constant, the firm cannot behave as a price-taker with output determined by (6). The following complement to Theorem 3 can be similarly proved.

THEOREM 4. *For an oligopolistic economy consisting of N firms that produce an homogeneous product, assume Firm X has a constant marginal cost c and adopts the following dynamic Walrasian strategy so that its output at period t is determined by*

$$x_t = x_{t-1} + g(\tilde{D}^{-1}(c) - \sum_{j=2}^{N} q_{t-h_j}^{(j)} - x_{t-1}), \tag{9}$$

where g is a monotonically increasing function with $g(0) = 0$, $h_j \geq 1$ is the information lag for firm j's output, and $q_{t-h}^{(j)}$, $j = 2, ..., N$, are the lagged outputs of other $N-1$ firms. When an intertemporal equilibrium is reached, we have the following:

i)Firm X makes normal profit at the equilibrium, so do any firm has positive equilibrium output;

ii)Any firm equipped with the same technology as Firm X but adopt traditional strategies such as Cournot best-response or Stackelberg leader strategy will be driven out the market in general due to loss of profit in equilibrium.

Proof. i) Let the equilibrium output vector be $(\bar{x}, \bar{q}_2, \bar{q}_3, ..., \bar{q}_N)$. Eq. (9) then implies

$$c = D(\bar{x} + \bar{q}_2 + ... + \bar{q}_N) \tag{10}$$

so that the profits for all firms having cost function $C(q) = cq$ must enjoy normal profits in equilibrium.

ii) Without loss of generality, assume that Firm 2 has the same cost function $C(q) = cq$ and adopts Cournot best-response (with or without knowledge of the exact outputs of other firms) so that its output y_t is determined from

$$D(x_{t-l_1} + y_t + \sum_{j=3}^{N} q_{t-l_j}^{(j)}) + D'(x_{t-l_1} + y_t + \sum_{j=3}^{N} z_{t-l_j}^{(j)} x) \cdot y_t = c, \tag{11}$$

where $l_j \geq 0$, $j = 1, 3, ..., N$ are the information lags for Firm Y.

When an intertemporal equilibrium $(\bar{x}, \bar{y}, \bar{q}_3, ..., \bar{q}_N)$ is reached, (11) simplifies to

$$D(\bar{x} + \bar{y} + \bar{q}_3 + ... + \bar{q}_N) + D'(\bar{x} + \bar{y} + \bar{q}_3 + ... + \bar{q}_N) \cdot \bar{y} = c. \tag{12}$$

[4]Needless to say, if duopoly game is just played with finite rounds (before reaching the equilibrium), then which firm has the relative profitability depends on the length of game as well as the strategies adopted by the sophisticated firm.

However, we know from (10) that $c = D(\bar{x} + \bar{y} + \bar{q}_3 + ... + \bar{q}_N)$ so that identity (12) suggests that

$$D'((\bar{x} + \bar{y} + \bar{q}_3 + ... + \bar{q}_N) \cdot \bar{y} = c,$$

that is, $\bar{y} = 0$ so long as $D' \neq 0$.

The proof for all other conventional Stackelberg leader strategy follow the same reasoning. ***Q.E.D.*** ∎

4.2. Static Walrasian strategy

We can infer directly from the relative-profitability regimes depicted in x-y plane that committing to the residual Walrasian residual equilibrium output ($x_t = q_w$) is itself a simplest relative-profitability strategy. We shall refer it as the *static Walrasian strategy*.

In a residual duopoly, define y_1 and y_2 as the solutions to $\pi^y(q_w, y_1) = 0$ and $\pi^x(q_w, y_2) = 0$ respectively. Referring to Figure 4.2, we see that when Firm X commits its output to q_w, the following inequalities exist between the profits of the two firms:

i) $\pi^x > \pi^y > 0$ when $y < y_1$,

ii) $\pi^x > 0 > \pi^y$ when $y_1 < y < y_2$,

iii) $0 > \pi^x > \pi^y$ when $y > y_2$.

The beauty of adopting static Walrasian strategy is more appreciated when the residual demand is linear.

PROPOSITION 5. ***(Efficiency Theorem)*** *When the residual demand function is linear,* $p = D(Q) = a - bQ$*, so long as* $C'' \geq 0$ *is satisfied, the static Walrasian strategy is the* efficient relative-profitability strategy *that maximizes the relative profit, regardless of the actual function form of* C*.*

Proof. Since the efficient relative-profitability strategy $x = R_r(y)$ is derived from (4), substituting $D(x+y) = a - b \cdot (x+y)$ into it yields

$$C'(x) = a - 2bx. \tag{13}$$

The efficient relative-profitability strategy is therefore to fix its output at a constant that is solved from (13).

However, the residual Walrasian equilibrium outputs $x_w = y_w = q_w$ is determined from

$$C'(q_w) = a - 2bq_w. \tag{14}$$

Comparing (14) with (13) leads to the conclusion. ***Q.E.D.*** ∎

Therefore, committing to static Walrasian strategy provides a firm with unbeatable commitment advantage. Given this information, its rival who is a profit maximizer, has no choice but to produce at $y^* = R_c(q_w)$, where R_c is the Cournot best response for Firm Y.

Apparently, the relative-profitability as well as the efficiency of static Walrasian strategy only holds true when the outputs of all other firms (whether they have same technology or not) are fixed.

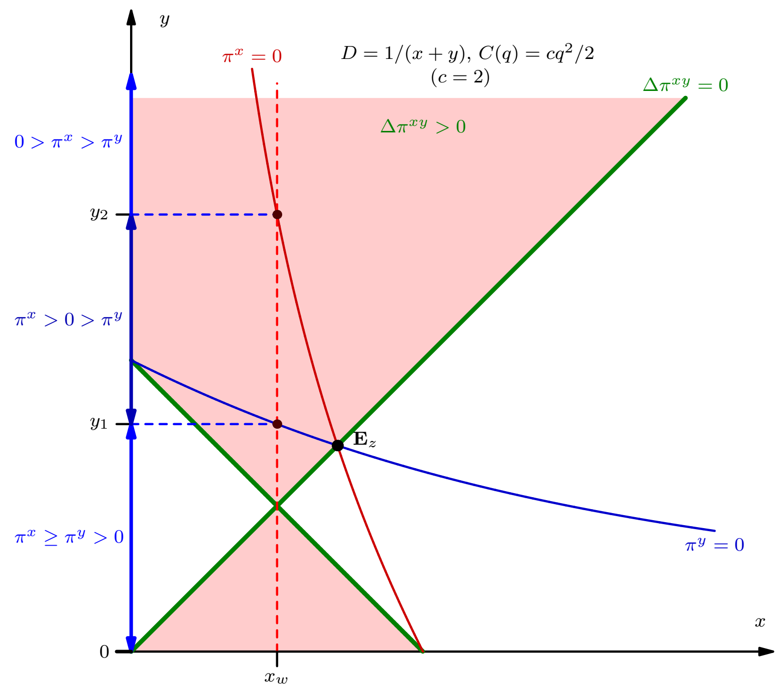


FIG. 4 Static Walrasian strategy

On the other hand, if all firms have identical cost C, then the static Walrasian strategy, i.e., producing at residual Walrasian equilibrium output amounts to producing at the competitive equilibrium level[5]. In fact, if we denote q_c as the competitive equilibrium output level so that

$$C'(q_c) = \tilde{D}(Nq_c)$$

then we can see that $q_c \equiv q_w$. Therefore, the relative profitability of static Walrasian strategy applies to the symmetric oligopoly where all firms have identical cost in the sense that

$$\tilde{D}((N-k)q_c + k\bar{q})q_c - C(q_c) > \tilde{D}((N-k)q_c + kq)\bar{q} - C(\bar{q}). \tag{15}$$

for all $\bar{q} \neq q_c$ and $1 \leq k \leq N$.

Inequality (15) was first provided in Vega-Redondo (1997), where it was shown that the competitive equilibrium q_c is a *global stable evolutionary strategy* in the sense that, starting with all firms producing at an identical q_c, if k firms change to defect and produce at another identical output level $\bar{q}$, they will definitely make less profit than those who remain in producing at q_c. Extending (15) to an n-residual oligopoly, by replacing N with $n < N$ and $\tilde{D}$ with D, we arrive at an analogous conclusion: the static Walrasian strategy (producing at q_w determined by $C'(q_w) = D(nq_w)$) is a global stable strategy from the evolutionary game theoretic point of view.

[5]Producing at the competitive level is commonly referred to as "Walrasian strategy" or "Walrasian behavior" directly in evolutionary game-theoretic literatures.

5. EVOLUTIONARY STABILITY OF DYNAMIC WALRASIAN STRATEGIES

For the static Walrasian strategy, in comparison to the dynamic Walrasian strategy, an noticeable short-coming is the former demands more information than the latter because to compute the residual Walrasian equilibrium output or competitive output, extra information such as (residual) demand and number of firms are needed. Moreover, the evolutionary stability of static Walrasian strategy is somewhat misleading. To see this, we should first note that concept of "strategy" adopted in either classical game theory or evolutionary game theory is a "static" concept and is more often defined as an option of action variable so that the strategies space consists of either a finite or infinite number of such choices[6]. A "strategy" in game-theoretic sense is said to be evolutionarily stable (ESS) if it, once adopted by all players, will not be discarded in favor of another "strategy" when a small fraction of players (mutants) choose another *single* different "strategy"[7], as implied by (15), where the relative-profit of producing at the competitive equilibrium output is checked against one mutant strategy (output level) $\bar{q}$ at a time. If more than one mutant strategies appear simultaneously, such relative profitability may disappear. In other words, if k_1 mutants produce at $\bar{q}_1$ and k_2 mutants produce at $\bar{q}_2$, then it may lead to the following opposite conclusion

$$\tilde{D}\left(\left(N-k_1-k_2\right)q_c+k_1\bar{q}_1+k_2\bar{q}_2\right)q_c-C\left(q_c\right)<\tilde{D}\left(\left(N-k_1-k_2\right)q_c+k_1\bar{q}_1+k_2\bar{q}_2\right)\bar{q}_i-C\left(\bar{q}_i\right)$$

$$i=1, \text{ or } 2.$$

Such examples can be easily constructed.

Example 3. Consider a symmetric oligopoly consisting of three firms, X, Y and Z, the output bundle is a vector of (x, y, z). Assume that the market inverse demand is given by $\tilde{D}(Q)=3/Q$, where $Q=x+y+z$ and that the identical cost is quadratic: $C(q)=q^2/2$ for $q>0$.

It follows from $\tilde{D}(3q_c)=C'(q_c)$ that the Walrasian equilibrium output level $q_c=1$.

Now $x\equiv q_c$, for any positive y and z, we have the profit ratio

$$\pi^x:\pi^y:\pi^z=5-y-z:y\left(6-y\left(1+y+z\right)\right):z\left(6-z\left(1+y+z\right)\right).$$

[6]Such usage is different from the much broader "strategy" understood in common sense. In particular, in other fields of economics, "strategy" is commonly refer to some kind of **behavioral rule** in response to changes resulted from rival's actions, market condition or external environment. In dynamic framework, it is usually expressed as a reaction function, such as the Cournot-best response (responding to rival's output or price), price-taking strategy and/or Cobweb strategy (responding to market price), adaptive and/or cautious-strategy (responding to unstable economy). This distinction is particularly clear for "Cournot-best response" that used both in game theory and in oligopolistic dynamics. In the former, all the quantities resulted from the response function are "strategies", while in the latter, the "best-reponse" itself is a strategy (like price-taking strategy), one of many "strategies" formed by different responses .

[7]The principal notions of evolutionary stability have evolved constantly. The idea of Evolutionarily stable strategies can be at least traced back to Ronald Fisher (1930). the formal definition were introduced by John Maynard Smith and George R. Price in a 1973 Nature paper, in which a strategy S is defined as an ESS if and only if, for all $T\neq S$, either i) $E(S,S)>E(T,S)$, or ii) $E(S,S)=E(T,S)$ and $E(S,T)>E(T,T)$, where T stands for any strategy and $E(\cdot,\cdot)$ is the expected payoff.

The first condition is sometimes called a *strict Nash equilibrium condition* or *equilibrium property* to indicate that the best strategy to face strategy S is also strategy S. The second is sometimes referred to as *Maynard Smith's second condition* or *stability property*, which emphasizes that if T does just as well against S as does S itself, then S will only be stable if it does better against T than T does against itself. In consequence, although the adoption of strategy T is neutral with respect to the payoff against strategy S, the population of players who continue to play strategy S have an advantage when playing against T.

There are many alternative definitions of ESS in different applications. They are not precisely equivalent to each other. For instance, Thomas (1984) changed the above two conditions to i) $E(S,S)\geq E(T,S)$, and ii) $E(S,T)>E(T,T)$ for all $T\neq S$.

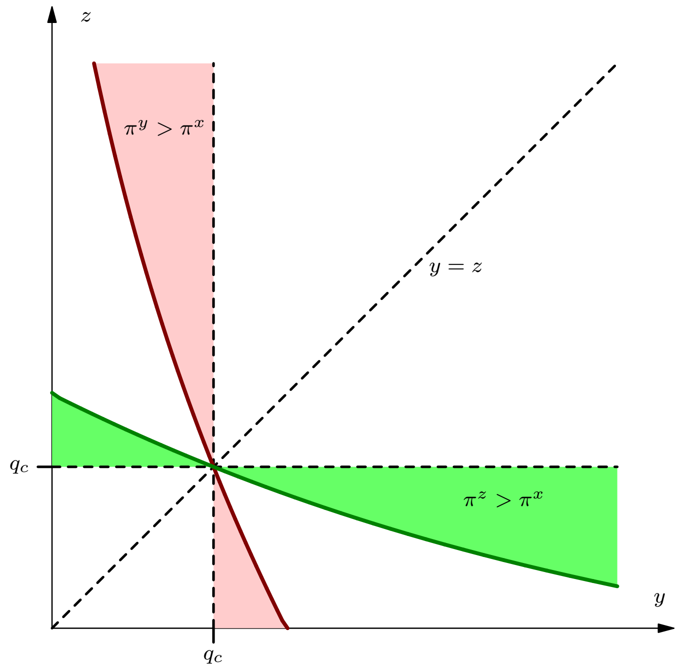


FIG. 5 Illustration of Example 3

We see that $\pi^y > \pi^x$ when $(1-y)\left(y^2+yz+2y-5+z\right) > 0$ and that $\pi^z > \pi^x$ when $(1-z)\left(z^2+yz+2z-5+y\right) > 0$. Therefore, there exist infinitely many (y,z) combinations such that make one of Firm Y and Z makes more profit than Firm X. Figure 5 illustrates the situation.

On the other hand, given any z, residual demand left for Firm X and Y is $D(Q) = 3/(Q+z)$ and the residual Walrasian equilibrium is obtained from the equilibrium condition $D(2q_w) = C'(q_w)$, which yields $q_w = (\sqrt{z^2+24} - z)/4$. Now for $x \equiv q_w$, we have

$$\pi^x - \pi^y = \frac{1}{32}\frac{(4y+z-\sqrt{z^2+24})^2(4y+z+3\sqrt{z^2+24})}{\sqrt{z^2+24}+3z+4y} \geq 0$$

for arbitrary y and z, where the equality holds if and only if $y = q_w$.

Suppose, instead, Firm X behaves as a price-taker, then for any given y and z, the equilibrium output $\bar{x}$ is obtained from the equilibrium condition $\tilde{D}(\bar{x}+y+z) = C'(\bar{x})$, which leads to $\bar{x} = (\sqrt{(y+z)^2+12} - y - z)/2$ and

$$\pi^x - \pi^y = \frac{1}{2}\frac{\left(3+y^2\right)\sqrt{(y+z)^2+12}+y^2(y+z)-(15y+3z)}{\sqrt{(y+z)^2+12}+y+z} \geq 0$$

where the equality holds if and only if $y = \bar{x} = q_w$. Similarly, $\pi^x \geq \pi^y$ and the equality holds if and only $z = \bar{x} = q_w$.

In additional to its weakness to against multiple mutants, static Walrasian strategy has another two shortcomings:

i) **Robustness issue**: according to Proposition 4, when the (residual) market demand changes or when entry or exit occur, the relative-profitability frontier shifts outward or inward, leading to the displacement of relative-profitability regime. Therefore, relative-profitability advantage in committing at the original Walrasian equilibrium output level may not be maintained;

ii) **Heterogeneous costs:** unlike the dynamic Walrasian strategy defined in Definition 7, the relative profitability of the static Walrasian strategy cannot be generalized to the non-symmetric oligopoly where firms have heterogeneous costs.

The implication between the dynamic and static Walrasian strategies can be seen straightforwardly by comparing (15) with the conclusions drawn in Theorem 3. When all firms have identical convex cost C, assume that k firms producing at arbitrary amounts $\bar{q}_1, \bar{q}_2, ...\bar{q}_k$, while remain $N-k$ firms produce at the dynamical equilibrium level x determined from

$$C'(\bar{x}) = \tilde{D}((N-k)\,\bar{x} + \sum_{j=1}^{k} \bar{q}_j) \tag{16}$$

then Theorem 3 implies that, for all $i = 1, 2..., k$, so long as $\bar{q}_i \neq x$, we have

$$\tilde{D}((N-k)\,\bar{x} + \sum_{j=1}^{k} \bar{q}_j)\bar{x} - C(\bar{x}) > \tilde{D}((N-k)\,\bar{q}_j + \sum_{j=1}^{k} \bar{q}_j)\bar{q}_i - C(\bar{q}_i), \tag{17}$$

The distinctions between (15) and (17) are apparent. However, the implications from (17) is far broader than the one from (15) in the sense that the latter can be logically inferred from the former either when "imitation" is assumed or when a payoff monotone evolutionary selection mechanism is enforced. As in the evolutionary game analysis where all firms seek to maximize relative profit, firms intend to imitate the successors by changing to the output levels that brings greater relative profit, (15) is then a long-run evolutionary outcome from (17) due to the fact that, along with the increasing number of "price-takers", the equilibrium output level $\bar{x}$ derived from (16) will approach the competitive equilibrium q_c. After all firms become price-takers, $\bar{x} = q_c$, any arbitrary deviation from q_c by any number of firms may promote short-term relative-profitability against the price-takers (here the price-takers will react to these deviation instead of sticking to q_c) but definitely ends up with a relative disadvantage status in terms of relative profit, should a new equilibrium be arrived. At this new equilibrium, the mutants may profit more or less than they do at the original equilibrium. Regardless of which case it is, in terms of the relative profitability, the mutants still performs worse than the "price-takers". In the long-run, by repeated imitations or selections, the competitive equilibrium is converged. The converse implication, that is, from (15) to (17), does not exist. For instance, for a duopoly, (15) refers to the relative-profitability of q_w while (17) refers to the relative profitability of f_w.

Just as the concepts of "static strategy" defined in the game theory should be distinguished with the "dynamic strategy" defined in our model, the distinctions between the concept of "strategy equilibrium" adopted in game theoretical analysis and the concept of "intertemporal equilibrium" adopted in dynamical analysis should also be noted. "Equilibrium" in game theory is interpreted in terms of static "strategies". A ESS is essentially a Nash-equilibrium with respect to the relative payoffs, which refers to the situation in which that you have no incentive to change your "strategy" (i.e., the output level in quantity-competition) when your rival sticks to his current "strategy". However, the "intertemporal equilibrium" that we concern here is a dynamic concept. It is nothing but a "steady state" resulted from mutual adjustments. When an intertemporal equilibrium is locally stable, if you change, your rival will react, and then the game will revert to whatever it was before. So it is not because you don't have incentive to change but because your rival always reacts properly to your change so as to make your change in vain and force you to adjust back to the original equilibrium status. The intertemporal equilibrium coincides with Nash-equilibrium if and only if all players adopted their respectively best-response strategies (such as Cournot reaction)[8].

Nevertheless, the conventional concept of evolutionary stability for static Walrasian strategy can be generalized to the dynamic Walrasian strategy as well. As we have emphasized repeatedly, except adopting Walrasian reaction directly, Walrasian strategy in general does not guarantee the relative profitability at each period of interaction. While an impatient imitator may not switch to the Walrasian strategy during the dynamical adjustment course, rational imitators will soon realize that it is always the long-run outcome that accounts and hence only imitate the strategy that can bring relatively higher profit when the system converges to a relatively stable state, i.e., a dynamical equilibrium. Along with the course of evolution, more and more firms become rational imitators and the Walrasian strategy evolves into an evolutionary stable strategy.

6. CONCLUSIVE REMARKS

We have proved that there exists a unique downward-sloping relative-profitability frontier, which together with 45-degree diagonal line, divides the output space into symmetrically located relative-profitability regimes for each firm. All relative-profitability reactions for a particular firm must pass through the residual Walrasian equilibrium, which happens to be the unique vertex that separates different relative-profitability regimes. Regardless of what strategies its rival may take, a firm behaving as a price-taker can achieve the relative profitability if an equilibrium is reached. Producing at the Walrasian output constantly, however, can bring about the relative profit at each and every period. Moreover, when the residual demand is linear, such commitment to invariant output turns out to be the efficient reaction in the sense that it unconsciously maximizes the relative profit against its rival.

These fundamental facts provide a direct economic interpretation and intuitive justification for the appreciation of Walrasian behavior from the different perspectives and provide a linkage between them. They do not only clarify the confusions and misunderstandings due to the different usages of same terminologies under different frameworks but also help to generalize the available results to a new level.

The relative profitability gained by a price-taker is conventionally interpreted as a consequence of betrayal and free-riding (Stigler 1950). It is well-known that in a cartel, any member has an incentive to increase its output above the agreed level so as to gain extra profit. The higher relative profit enjoyed by a betrayer (a price-taker) is achieved through hurting those who remain in the collusion more than hurting itself[9]. In other words, the price paid for betraying is the reduction in instantaneous absolute profit. Such observations lead to the questioning of the rationale of price-taking strategy, in particular for the economists who believe the absolute profitability is what a firm should concern.

[8]The price-taking reaction can be regarded as a best-response strategy in boarder sense: it is the reaction best-response to the price instead to the outputs of rivals.

[9]This behavior is described as "spiteful" in Schaffer (1989).

However, when the number of firms in an oligopoly is relative large, a firm may prefer to behave as a "price-taker" not just for the higher relative profit compared to the rest but also for higher instantaneous profit. In the terminology of game-theory, the "price-taking" can be a dominant strategy for some firms when the oligopoly is composed of price-takers and sophisticated firms that adopt Cournot best-response strategy and the price-takers. Such observation is first revealed in Huang (2002) and will be further explored in the second part of this research, where the advantage of price-taking strategy against conventional sophisticated strategies such as Cournot best-response and collusion will be further explored.

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8. APPENDIX A: PROOFS

Proof of Theorem 1

Without loss of generality, we proceed with the case in which $y < x$ since the proof for the case in which $y > x$ follows directly due to the symmetry of x and y.

Let $y = f_w(x)$ be the inverse function of the Walrasian reaction defined in (1). This implies that the curve on x-y plane satisfies the following identity:

$$D(x + f_w(x)) = C'(x).$$

First, we shall show that $f_e(x) \geq f_w(x)$ for all $x \in Q$ and the equality holds only when $f_e(x) = x$. In other words, given any level of y, f_e lies on the right-hand side of f_w.

Notice that by the definition of $\Delta^{xy}(x, f_e(x)) = 0$, we have

$$\Delta^{xy}(x, f_e(x)) = (x - f_e(x)) D(x + f_e(x)) - (C(x) - C(f_e(x))) = 0.$$

So $f_e(x)$ has an implicitly defined solution of

$$f_e(x) = x - \frac{C(x) - C(f_e(x))}{D(x + f_e(x))}$$

Or, equivalently, $D(x + f_e(x)) = (C(x) - C(f_e(x))) / (x - f_e(x))$.

With the convexity of C, we have

$$\frac{C(x) - C(f_e(x))}{x - f_e(x)} \leq C'(x) \text{ for } x \geq f_e(x),$$

where the equality holds only when $f_e(x) = x$, therefore

$$D(x + f_e(x)) \leq C'(x) = D(x + f_w(x)).$$

The downward-sloping property of D implies that $f_w(x) \leq f_e(x)$, with the equality being held when $f_e(x) = x$. Therefore, for a given x and $f_e < x$, we have f_e lies above f_w. This is equivalent to saying that f_e lies on the right-hand side of f_w for any given y.

To prove the uniqueness of f_e, we first show that for any fixed y, $\Delta^{xy}(x, y)$ is a monotonically decreasing function of x for $x > f_w^{-1}(y)$.

Given a fixed x, we have $D(x + y) \lesseqgtr C'(x)$ when $y \gtreqless f_w(x)$, or, equivalently, given a y, $D(x + y) \lesseqgtr C'(x)$ when $x \gtreqless f_w^{-1}(y)$. However, for any fixed y, we have

$$\frac{\partial \Delta^{xy}(x, y)}{\partial x} = D - C'(x) + (x - y) D'.$$

Due to $x > y$ (by assumption) and the fact that $D(x + y) \lesseqgtr C'(x)$ for $x \gtreqless f_w^{-1}(y)$, we know that

$$\frac{\partial \Delta^{xy}(x, y)}{\partial x} < 0 \text{ when } x > f_w^{-1}(y).$$

This implies that $\Delta^{xy}(x,y)$ is a monotonically decreasing function of x for $x > f_w^{-1}(y)$. Therefore, for any fixed y, when x is greater than the value implied by the curve $x = f_e^{-1}(y)$ ($\Delta^{xy}(x,y) = 0$), $\Delta^{xy}(x,y)$ is always negative. Since it is impossible to have $\Delta^{xy}(x,y) = 0$ for $x > f_e^{-1}(y)$, uniqueness of f_e is ensured.

ii) The downward sloping characteristics of f_e follows straightforwardly from the uniqueness of f_e. If not, there must exist a x_0 such that $f_e'(x_0) = 0$ with $f_e''(x_0) > 0$. This in turn implies the existence of a $\epsilon \to 0^+$ such that for $y_\epsilon = f_e^{-1}(x_0) + \epsilon$, $f_e^{-1}(y_\epsilon)$ has multiple values, a contradiction.

iii) The intersection of the trivial equal profitability curve and the relative-profitability frontier gives rise to $x = y = q_w$. The rest of conclusions are obvious. ***Q.E.D.***

Proof of Proposition 1

Proof. (2) is obtained by rearrangement after taking derivative with respect to x over both sides of the identity

$$D(x + f_e)(x - f_e) = C(x) - C(f_e).$$

Consider the segment of $y = f_e(x)$ below the 45-degree line, we have $y < x$ and $C'(x) \geq D > C'(y)$, which implies that $f_e' < 0$ for $y < x$. In particular, when $x \to q_w$ and $f_e \to q_w$, we have $C'(x) - D \to 0$, so that

$$\lim_{x \to q_w,\ y \to q_w} f_e' = \frac{C'(x) - D - (x - f_e) D'}{C'(f_e) - D + (x - f_e) D'} = \lim_{x \to q_w,\ y \to q_w} \frac{C'(x) - D}{C'(f_e) - D} = -1.$$

Q.E.D. ∎

Proof of Proposition 3

Proof. For an arbitrary D that satisfies $D(0) > b$, when $C(q) = a + bq + cq^2$ and $y = f_e(x)$, we have

$$\Delta^{xy}(x,y) = D(x+y) \cdot (x-y) - b \cdot (x-y) - c \cdot \left(x^2 - y^2\right) = 0.$$

Therefore, for $x \neq y$, we have

$$D(x+y) - c \cdot (x+y) = b \tag{18}$$

which implies that

$$D(x+y) = \hat{D}^{-1}(b)$$

where $\hat{D}^{-1}$ is the inverse function defined by $\hat{D}(z) \doteq D(z) - cz$. Therefore, the equal-profitability curve f_e is always linear, regardless of the residual demand D.

When $c = 0$, (18) simplifies to

$$D(x+y) = b = C'(x) = C'(y)$$

which is nothing but the Walrasian reaction curve for both Firm X and Y. It is also the normal-profitability curves for both firms due to zero fixed cost. ***Q.E.D.*** ∎

Proof of Proposition 4

Proof. i) For a given x, denote $C^{(2)}(x) = (1+\alpha) C^{(1)}(x)$. When $\alpha \neq 0$, the equal-profitability curve $f_e(\alpha, x)$ is implicitly defined by

$$(x - f_e(\alpha, x)) D(x + f_e(\alpha, x)) = (1+\alpha)(C(x) - C(f_e(\alpha, x))). \tag{19}$$

Taking derivative with respect to α over both sides of (19) and rearranging yields

$$\begin{aligned}\frac{\partial f_e(\alpha, x)}{\partial \alpha} &= \frac{C(x) - C(f_e(\alpha, x))}{(1+\alpha) C'(f_e(\alpha, x)) + (x - f_e(\alpha, x)) D' - D} \\ &= \frac{C(x) - C(f_e(\alpha, x))}{(1+\alpha)\left(C'(f_e(\alpha, x)) - \dfrac{C(x) - C(f_e(\alpha, x))}{x - f_e(\alpha, x)}\right) + (x - f_e(\alpha, x)) D'}\end{aligned}$$

Since the convexity of C implies that

$$\frac{C(x) - C(f_e(\alpha, x))}{x - f_e(\alpha, x)} \gtreqless C'(f_e(\alpha, x)) \text{ when } x \gtreqless f_e(\alpha, x),$$

we thus have $\dfrac{\partial f_e(\alpha, x)}{\partial \alpha} < 0$ for all α.

ii) Since for any given $z = x+y$, we are able to define a $\beta(z)$ such that $D^{(2)}(z) = (1+\beta(z)) D^{(1)}(z)$, the proof can be accomplished by showing that f_e, which is a function of $\beta(z)$, shifts rightwards with increasing $\beta(z)$ when the residual demand function is given by $(1+\beta) D(z)$. Now the equal-profitability curve $f_e(\beta, x)$ is implicitly defined by

$$(x - f_e(\beta, x))(1+\beta) D(x + f_e(\beta, x)) = C(x) - C(f_e(\beta, x)). \tag{20}$$

Taking derivative with respect to β over both sides of (20) and rearranging would suggest that

$$\frac{\partial f_e}{\partial \beta} = \frac{(x - f_e(\beta, x)) D(x + f_e(\beta, x))}{\dfrac{C(x) - C(f_e(\beta, x))}{x - f_e(\beta, x)} - C'(f_e(\beta, x)) - (x - f_e(\beta, x))(1+\beta) D'} > 0$$

for all β. ***Q.E.D.*** ∎

Proof of Proposition 9

Proof. Verification of i) and iii) are straightforward by noticing that when (21) is satisfied, the profit from Firm X is given by $\pi^x(x, f_w(x)) = C'(x) x - C(x)$ and $\Delta^{xy}(x, y)|_{y=f_w(x)} = C'(x)(x-y) - (C(x) - C(f_w(x)))$.

ii) Since $y = f_w(x)$ and $\pi^y(x, f_w(x)) = C'(x) f_w(x) - C(f_w(x))$, we thus have

$$\frac{d\pi^y(x, y)}{dx}|_{y=f_w(x)} = C''(x) y + (C'(x) - C'(y)) f_w'(x),$$

substituting $f_w'(x) = C''(x)/D' - 1$ into which yields

$$\frac{d\pi^y(x, y)}{dx}|_{y=f_w(x)} = \frac{1}{D'}(yC''(x) D' + (C'(x) - C'(y))(C''(x) - D')).$$

Therefore so long as $x \le q_w$ $(x \le y)$ we have $\dfrac{d\pi^y(x,y)}{dx} > 0$. Since $\pi^y(q_w, q_w) > 0$ and $\pi^y(x^*, 0) = 0$, π^y must have at least one local maximum between q_w and $x^* = R_w(0)$. One of such local maximum occurs at $x_s > q_w$ with $\dfrac{d\pi^y(x_s, y_s)}{dx}|_{y_s=f_w(x_s)} = 0$, that is, Eq. (22).

Moreover it can be verified that

$$\frac{d^2\pi^y(x,y)}{dx^2}|_{y=f_w(x)} = D'C''(x)\left(2C''(x) - 2D' - yD''\right) - C''(y)\left(C''(x) - D'\right)^2 + \left(D'\left(C'''(x) + D''\right) - 2D''C''(x)\right)\left(C'(x) - C'(y)\right)$$

which is negative for all $y = f_w(x)$ when conditions (23) are satisfied. ***Q.E.D.*** ∎

Proof of Proposition 2

Proof. Notice that R_w, R_r and R_c meet the following identities, respectively.

$$C'(R_w(y)) = D(R_w(y) + y),$$
$$C'(R_r(y)) = D(R_r(y) + y) + (R_r(y) - y)D'(R_r(y) + y),$$
$$C'(R_c(y)) = D(R_c(y) + y) + R_c(y)D'(R_c(y) + y).$$

It is easy to see that for any given y, we have $R_c(y) < R_w(y)$ and $R_c(y) < R_r(y)$. However, $R_r(y)$ and $R_w(y)$ intersect at $x = y = q_w$. Moreover, as can be seen from Figure 6 that, for a given y, the function of $G(x) \doteq D(x+y) + (x-y)D'(x+y)$ lies above (below) $D(x+y)$ for $x < y$ $(x > y)$, which implies that $R_r(y)$, the intersection of G and C', is greater (less) than $R_w(y)$ if $x < y$ $(x > y)$. ***Q.E.D.*** ∎

9. APPENDIX B: ISO-PROFIT CURVE AND RELATIVE-PROFITABILITY FRONTIER

In this appendix, we shall reveal some relations among iso-profit curve, the normal-profit curves and the equal-profitability curve. The information revealed is important for the analysis of interactions of various strategies adopted by the duopolistic firms.

DEFINITION 8. *Iso-profit curve and normal-profit curve for Firm i*

Iso-profit curve for Firm i is the curve in the x-y plane that depicts the relationship between x and y so that the profit for Firm i is fixed at a constant level $\pi^i(x,y) = \pi_0$, where π_0 is a constant. In particular, when $\pi_0 = 0$, the iso-profit curve will be referred to as the *normal-profitability curve.*

Let q_w be the residual Walrasian equilibrium output such that $D(2q_w) = C'(q_w)$ and denote $E_w = (q_w, q_w)$ as the residual Walrasian equilibrium point in the x-y plane. Furthermore, let q_n represents the maximum q such that $\pi^x(q_n, q_n) = \pi^y(q_n, q_n) = 0$, that is, q_n satisfies the identity

$$q_n D(2q_n) = C(q_n).$$

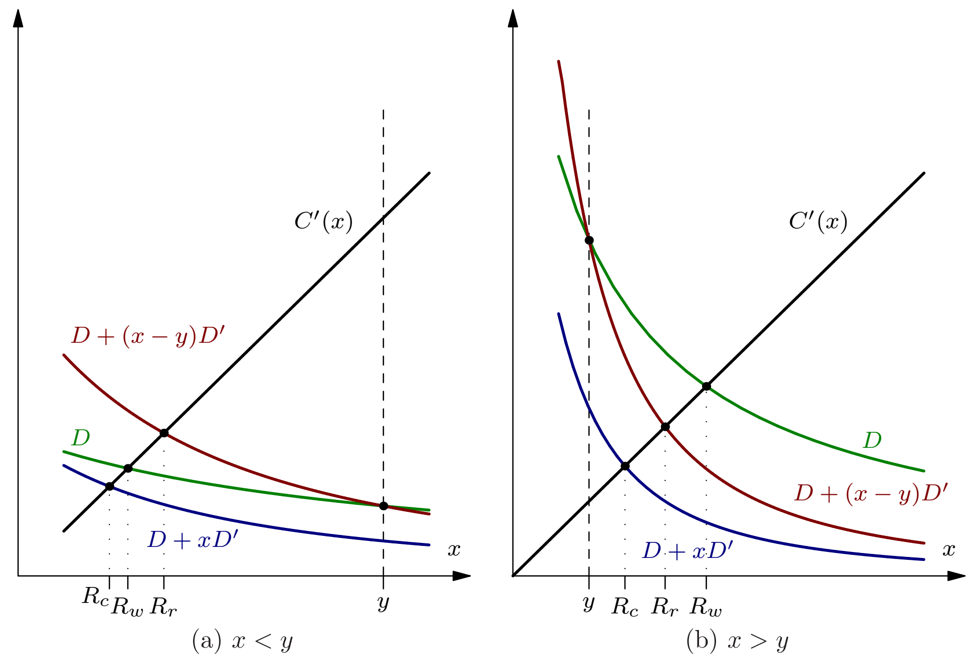


FIG. 6 Relative magnitudes of R_r, R_c and R_w

The analytical relationship between iso-profit curve and equal profitability curves is summarized in the following proposition:

PROPOSITION 6. *i) Along the 45-degree line, only at residual Walrasian equilibrium (q_w, q_w) will the slope of iso-profit curves of both firms equal -1. In other words, at the residual Walrasian equilibrium (q_w, q_w), the iso-profit curves (for both firms) are tangent to the equal-profitability curve.*

Proof. For the iso-profit curve of Firm X, we have

$$\pi^x(x,y) = xD(x,y) - C(x)$$

The slope of $\pi^x(x,y)$ is given by

$$D(x,y) + xD'(x,y)\left(1 + \frac{\partial y}{\partial x}\right) - C'(x) = 0$$

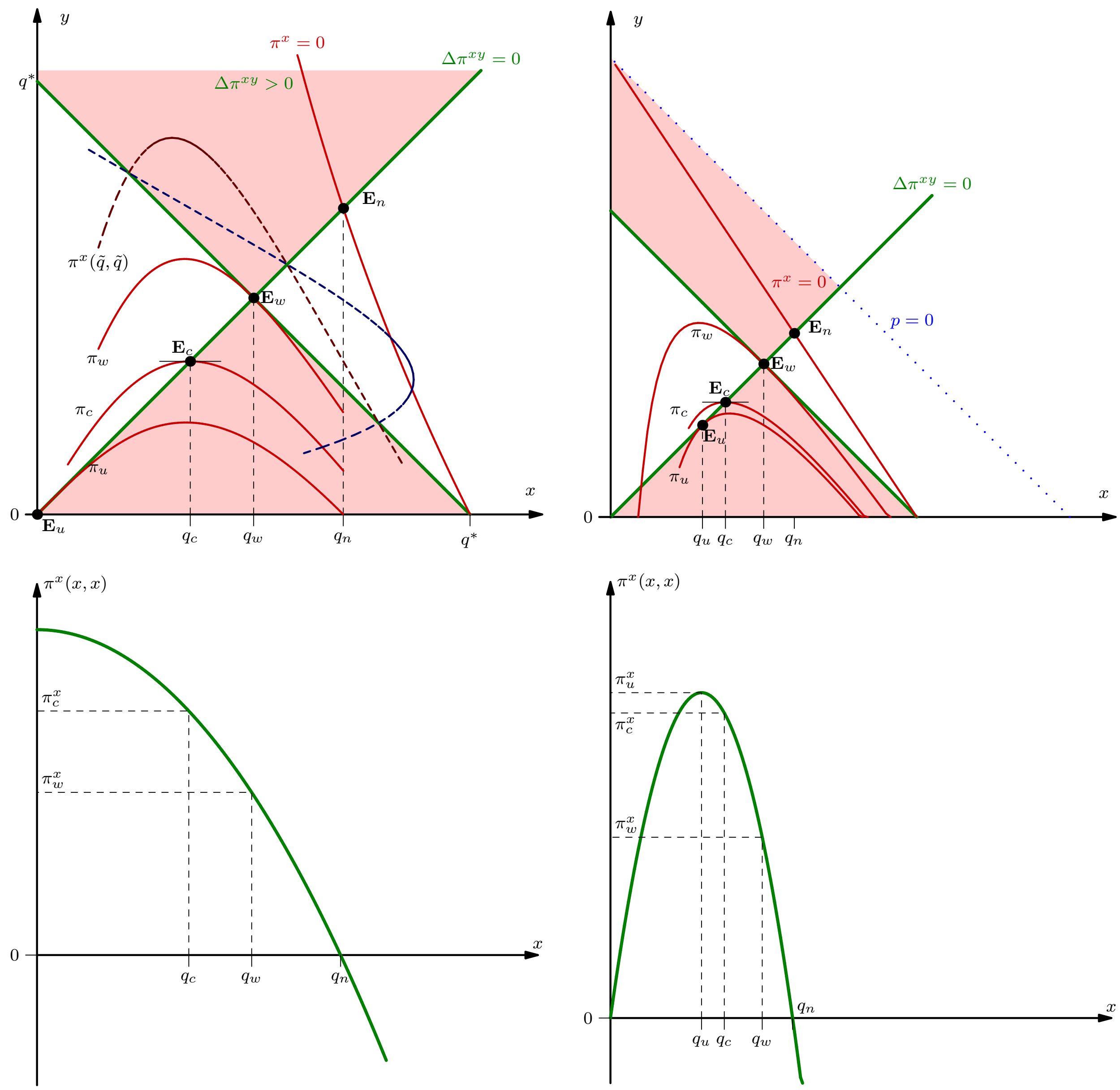
or,

$$\frac{\partial y}{\partial x} = \frac{C'(x) - D}{xD'(x,y)} - 1.$$

Since $y = x$ along the 45-degree line, we thus have $\partial y/\partial x = -1$ if and only if $C'(x) = D$, i.e., if and only if at the residual Walrasian equilibrium. ***Q.E.D.*** ∎

When $x \in (q_w, q_n)$, the profit is positive. The iso-profit curve will intersect twice with the equal-profitability curve f_e. The following proposition indicates that these two intersection points are anti-symmetric.

PROPOSITION 7. *If $q_n > q_w$, then for any $\tilde{q} \in (q_w, q_n)$, there exists a pair $(q_1, f_e(q_1))$, $(f_e(q_1), q_1)$ on*



(a) Iso-elastic demand $D(Q) = 1/Q$

(b) Linear demand $D(Q) = 1 - Q$

FIG. 7 Iso-profit and Equal-Profitability curves ($C(q) = q^2/2$)

the equal profitability curve $y = f_e(x)$ *such that*

$$\Delta^{xy}(q_1, f_e(q_1)) = \Delta^{xy}(f_e(q_1), q_1) = \Delta^{xy}(\tilde{q}, \tilde{q})$$

and

$$\pi^i(q_1, f_e(q_1)) = \pi^i(f_e(q_1), q_1) = \pi^i(\tilde{q}, \tilde{q}) \text{ for } i = x, y.$$

Proof. As depicted in Figure7.(a), the normal-profit curve $\pi^x(x, y) = 0$ passes (q^n, q^n), $(q^*, 0)$ and $(0, q^*)$ three points.

For any $q^n > \tilde{q} \geq q_w$, $\pi^i(\tilde{q}, \tilde{q}) > 0$, we are able to identify a q_1 on the lower segment of $y = f_e(x)$ such that $\pi^x(q_1, f_e(q_1)) = \pi^x(\tilde{q}, \tilde{q})$. By the symmetricity of $\pi^x(x, y)$ and $\pi^y(y, x)$ and the anti-symmetricity of $\Delta^{xy} = 0$, we must have

$$\pi^x(q_1, f_e(q_1)) = \pi^y(f_e(q_1), q_1) = \pi^x(f_e(q_1), q_1)$$

where the first equality follows from the symmetricity of $\pi^x(x, y)$ and $\pi^y(x, y)$ and the second one from $\Delta^{xy} = 0$. ***Q.E.D.*** ∎

Remark 3. The profits levels along $y = f_e(x)$ thus ranges from 0 to π_w. Unless $y = x$, any strategy that results in $y \neq x$ and consequently $\Delta^{xy} = 0$ results in a profit that is less than the Walrasian profit.

To see the relationship between iso-profitability curve and iso-profits curves for the case of residual Walrasian equilibrium, Cournot equilibrium and collusive equilibrium, we let q_c denote the identical Cournot equilibrium output when both firms adopt Cournot strategy where q_c satisfies the following identity:

$$D(2q_c) + q_c D'(2q_c) = C'(q_c),$$

then it can be shown that

$$\frac{\partial \pi(x, x)}{\partial x}|_{x=q_c} = 0.$$

On the other hand, denote q_u as the average collusive output when both firms collude as a Cartel, where q_u is determined from

$$D(2q_u) + 2q_u D'(2q_u) = C'(q_u),$$

then it can be verified that the iso-profit curve is tangent to the 45degree line at q_u (at which $\pi(x, x)$ reaches its maximum value).

Figure 7 (a) and (b) illustrate the relationship between equal-profitability curve and iso-profits curves for the case of residual Walrasian equilibrium, Cournot equilibrium and collusive equilibrium in x-y plane for the iso-elastic residual demand ($D(Q) = 1/Q$) and the linear residual demand $D(Q) = 1 - Q$, respectively. To see how profit level changes along the 45-degree line, the profit functions are depicted underneath the respective graphs.

Notice that, when there is no fixed cost, the intercepts of the equal-profitability curve $y = f_e(x)$ at x and y axes, denotes as x^* and y^* respectively, satisfy the following identities:

$$D(x^*) = C(x^*)/x^*, D(y^*) = C(y^*)/y^*,$$

and consequently, we have $\pi^x = \pi^y = 0$ at these two extreme points. In other words, we know the normal-profit curve for X (Y) must intersect at x^* (y^*).

10. APPENDIX C: CHARACTERIZATION OF WALRASIAN REACTION

In a residual-duopoly in which Firm X exercises the Walrasian strategy, regardless of whether the production technology exhibits constant return or decreasing return to scale, Walrasian reaction defined by (1) is where an equilibrium outcome $(\bar{x}, \bar{y})$ lies on. The following proposition characterizes the shape of Walrasian reaction curve.

PROPOSITION 8. *When $D' < 0$ and $C'' \geq 0$ are satisfied, the inverse of the Walrasian reaction $y = f_w(x)$ implicitly defined by*

$$D(x + f_w(x)) = C'(x) \tag{21}$$

has the following properties[10]*:*

i) f_w is a downward sloping curve with a slope greater than unity in absolute value, i.e.,

$$f_w'(\bar{x}) = C''(\bar{x})/D' - 1 < -1;$$

ii) $f_w''(\bar{x}) = \frac{-1}{D'}\left((C''(\bar{x})/D')^2 D'' - C'''(\bar{x})\right)$, which is positive if $D'' \geq 0$ and $C''' \leq 0$.

Proof. Direct verification. ∎

Remark 4. $|f_w'(\bar{x})| > 1$ implies that when x increases, the reduction of y exceeds the increment of x so that the industrial output decreases and the market price increases.

Remark 5. Conditions $D'' \geq 0$ and $C''' \leq 0$ are just sufficient but not necessary conditions for the convexity of f_w. For instance, when $D = 1 - Q$ and $C(q) = q^3$, we have $C'''(q) > 0$ but $f_w''(x) = 2x^{-4} > 0$. The graph of f_w is depicted in Fig. 2.(c).

Since we are more interested in the relative change of profits and the relative profits along the Walrasian reaction curve, we have

PROPOSITION 9. *Along the Walrasian reaction curve whose inverse is given by $y = f_w(x)$, we have*

i) π^x is a monotonically increasing function of x with $\frac{d\pi^x}{dx}|_{y=f_w(x)} = xC''(x)$ so that π^x achieves its maximum at $x^* = R_w(0)$;

[10] For the Walrisian reaction, we have

$$R_w'(y) = D'/\left(C''(\bar{x}) - D'\right)$$

and

$$R_w''(y) = \frac{D''(C''(\bar{x}))^2 - (D')^2 C'''(x)}{(C''(x) - D')^3}.$$

ii) π^y *is a monotonically increasing function of* x *when* $x < y$ *(i.e,* $x < q_w$*) and it achieves a local maximum at* $x_s > q_w$ *that is implicitly defined by*

$$\left(C'\left(x_s\right) - C'\left(f_w\left(x_s\right)\right)\right)\left(C''\left(x_s\right) - D'\right) = -f_w\left(x_s\right) D' C''\left(x_s\right). \tag{22}$$

Moreover, if

$$D' + D''y < 0,\ D'' > 0 \text{ and } C''' < 0, \tag{23}$$

then x_s *is the unique maximum.*

iii) $\dfrac{d\Delta^{xy}\left(\bar{x},\bar{y}\right)}{dx}|_{y=f_w(x)} = C''\left(x\right)\left(x-y\right) - \left(C'\left(x\right) - C'\left(y\right)\right) f_w'$ *so that*

$$\frac{d\Delta^{xy}\left(\bar{x},\bar{y}\right)}{dx}|_{y=f_w(x)} \gtreqless 0 \text{ if } x \gtreqless q_w).$$

The relative profit between two firms, however, is a monotonically increasing and decreasing function of x when x is greater or less than q_w, respectively which is consistent with the fact that the relative profit difference reaches to its minimum (zero) at the residual Walrasian equilibrium E_w.

Finally, it is worthwhile to point out that for a symmetric duopoly with Firm X exercising Walrasian strategy, a local maximum profit is achieved when Firm Y behaves as a quasi-Stackelberg leader in the way that the Walrasian reaction curve R_w is known to Firm Y and is taken into consideration in its profit maximization strategy. Instead of proceeding to the relevant proof, we introduce a more general result related to strategically related parameter.

PROPOSITION 10. ***Comparative statics for a strategic parameter*** γ_i

When $D' < 0$ *and* $C'' > 0$ *is satisfied, for an equilibrium established by (21) and*

$$D - h\left(\left\{\gamma_i\right\}, \bar{x}, \bar{y}\right) = C'\left(\bar{y}\right) \tag{24}$$

where $\{\gamma_i\}$ *is a set of strategic parameters that do not appear in both* D *and* C*, then we have*

$$\frac{\partial \bar{x}}{\partial \gamma_i} = -\frac{D'}{\bar{H}}\frac{\partial h}{\partial \gamma_i} \tag{25}$$

$$\frac{\partial \bar{y}}{\partial \gamma} = \frac{D' - C''\left(\bar{x}\right)}{\bar{H}}\frac{\partial h}{\partial \gamma_i} \tag{26}$$

where

$$\bar{H} \doteq C''\left(\bar{x}\right) C''\left(\bar{y}\right) - D'\left(C''\left(\bar{x}\right) + C''\left(\bar{y}\right)\right) + \left(\partial h/\partial x - \partial h/\partial y\right) D', \tag{27}$$

is nonnegative if

$$\partial h/\partial x \le \partial h/\partial y. \tag{28}$$

As an application of Proposition 10, we examine the traditional reaction by Firm Y that can be unified with conjectural variation formulation. In general, assume that Firm Y's reaction curve is implicitly defined by

$$D + \left(\alpha + \beta R_w'\left(y\right)\right)\bar{y}D' = C'\left(\bar{y}\right)$$

with $\alpha \ge \beta \ge 0$.

While Cournot best-response corresponds to the case in which $\alpha = 1$ and $\beta = 0$, Stackelberg-alike strategy is characterized by $\alpha = \beta = 1$.

For fixed α and β, let $h(\alpha, \beta, x, y) = -\bar{y} D'(\bar{x} + \bar{y})(\alpha + \beta R'_w(y))$, then we have $h(\alpha, \beta, x, y) > 0$ if $0 \leq \beta < \alpha$ (due to $|R'_w(y)| < 1$).

$$\partial h / \partial x - \partial h / \partial y = [\overbrace{\alpha + \beta R'_w(y)}^{(+)} + \beta \bar{y} R''_w(y)] D'.$$

It follows from Proposition 8 that if $D'' > 0$ and $C''' < 0$, we have $R''_w(y) > 0$, which in turn implies that $\partial h / \partial x < \partial h / \partial y$.

Due to the facts that

$$\frac{\partial h(\alpha, \beta, x, y)}{\partial \alpha} = -\bar{y} D' > 0, \frac{\partial h(\alpha, \beta, x, y)}{\partial \beta} = -\bar{y} D' R'_w(y) < 0,$$

for $\beta < \alpha$ and

$$\frac{\partial h(\alpha, \beta, x, y)}{\partial \alpha} = -\bar{y} D'(1 + R'_w(y)) > 0$$

for $\alpha = \beta$, we are able to get the following comparative statics immediately from Theorem 9 and Proposition 10:

	$\beta < \alpha$		$\alpha = \beta$
	$\partial / \partial \beta$	$\partial / \partial \alpha$	$\partial / \partial \alpha$
x	$-$	$+$	$+$
y	$+$	$-$	$-$
π^x	$-$	$+$	$+$
$\Delta \pi^{xy}$	$-$	$+$	$+$

In particular, for $\alpha = \beta = 1$, substituting D with $C'(x)$ in (24) and rearrange gives us

$$C'(\bar{x}) + \bar{y} D'(1 + \frac{D'}{C''(\bar{x}) - D'}) = C'(\bar{y})$$

which is nothing but the first-order maximization condition for π^y given by (22). That is to say that Stackelberg alike strategy (taking into account the reaction curve R_w) leads to the maximum profit for Firm Y, which is at least true when Walrasian reaction curve is convex.