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AUTARKIC INDETERMINACY AND TRADE DETERMINACY†

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Abstract. We extend the model of Nishimura and Shimomura (2002) to consider a two-country framework where under autarky indeterminacy arises in one country but determinacy in the other, and show that indeterminacy could be eliminated when trade takes place between the two.

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1 Introduction

By now, it is well-known that indeterminacy of the equilibrium path can arise in an optimal growth framework when externalities are introduced into production. Seminal work by Benhabib and Farmer (1994, 1996) demonstrates that externalities-generated increasing returns to scale may give rise to indeterminacy. More recently, Benhabib and Nishimura (1998) and Benhabib, Meng and Nishimura (2000) prove in multi-sector models that constant returns to scale production possessing mild externalities may also cause indeterminacy to arise. Importantly, they have heightened interest in the existence of endogenous fluctuations in the macroeconomics literature.

This paper examines whether indeterminacy or uniqueness of the equilibrium can be "transmitted" across countries.[1] In particular, we ask what the eventual steady state properties of a two-country world would be if trade is introduced between a country that features indeterminacy under autarky and a country that does not. We believe this question has important implications for the empirical literature that is largely closed economy oriented: while indeterminacy may be empirically observed in closed economy analysis, both determinacy and indeterminacy are possible outcomes when trade takes place.

Our study is based largely on the seminal work of Nishimura and Shimomura (2002), who examine a two-country Heckscher-Ohlin model under sector-specific externalities.[2] In our exercise, we break the symmetry in which externalities enter the production function in the two countries. The simplest way to do so is to remove the sector-specific externalities in one country while keeping them in the other country consistent with the original framework. By setting up our problem in this manner, we obtain a starting point where determinacy holds under autarky in the first country and indeterminacy in the second. We then ask whether indeterminacy can be eliminated when trade is engaged with the country having a determinate equilibrium under autarky, or conversely, whether the initially saddle-path stable country would end up importing indeterminacy instead.

A technical issue arises with our modeling assumption however - the integrated equilibrium is no longer supportable since returns to private factors always differ across countries. Our solution proceeds by imposing that factors of production are internationally immobile. The clear benefit of doing so is to minimize the interation between the two countries, that is, to keep each country as "separate" as possible, so as to isolate the effect of trade on the dynamic properties in the model. The main finding of our experiment is that trade may overturn local indeterminacy that previously exists under autarky. Understanding that many outcomes are possible in the class of models we are working on, including indeterminacy under trade, we wish to highlight our special case as it will

[1] Although their focus is not on indeterminacy, Nishimura and Yano (1993) are among the very first to theoretically examine how a country's pre-trade capital accumulation pattern may coincide or not with its and the world's aggregate post-trade patterns, which in fact is an investigation on the transmission of capital accumulation patterns via trade.

[2] Nishimura and Shimomura (2002) show that when private and social factor intensity rankings are the reverse of the other, indeterminacy of the equilibrium path in the world market is possible. Their seminal model assumes indeterminacy in each country before trade while we change that assumption in our analysis.

complement the seminal findings of Nishimura and Shimomura (2002) and others in the literature.

Our main result may be compared to some recent and key contributions. Specializing the model in Nishimura and Yano (1993) to Cobb-Douglas technologies, Nishimura, Venditti and Yano (2006a) show that endogenous two-period cycles at the world level may be derived from endogenous fluctuations existing in one country under autarky despite the autarkic equilibrium path of the trading partner is monotonically convergent. In an extension to include sector-specific externalities in a two-country, two-good, two-factor model, Nishimura, Venditti and Yano (2006b) show that market integration with a country exhibiting constant returns at the social level and local indeterminacy may have destabilizing effects for some countries. They have assumed capital to be internationally mobile while we have immobility to focus on the effect of trade. It would be interesting to extend our work to incorporate capital mobility in the future for further investigation and comparison.

For the rest of the paper, Section 2 shows that the uniqueness of the equilibrium path can exist when trade takes place between two countries, one with a locally indeterminate equilibrium under autarky and the other with a locally determinate equilibrium. Section 3 concludes.

2 The Model

The model is outlined as follows. Consider the home country first, which is populated by an infinitely-lived representative agent having an instantaneous utility of

$$U(C) = \frac{C^{1-\eta}}{1-\eta}$$

where C denotes the home country's consumption and $1/\eta \in (1, \infty)$ represents the intertemporal elasticity of substitution in consumption. Let both consumption good and investment good, with the latter given by I, be produced with the Cobb-Douglas technology. The household's intertemporal problem is to maximize

$$\underset{\mathrm{C}}{\mathrm{Max}} \int_0^\infty \frac{C^{1-\eta}}{1-\eta} e^{-\rho t} dt \tag{1}$$

subject to

$$\hat{I} = L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} \quad \text{and} \quad \hat{C} = L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{2}$$

$$K = K_I + K_C \quad \text{and} \quad L = L_I + L_C \tag{3}$$

$$\dot{K} = \hat{I} + p\hat{C} - \delta K - pC \tag{4}$$

$$K(0) \quad \text{given}$$

where p is the price of consumption in terms of investment, $\rho \in (0, \infty)$ is the subjective rate of time discount, and $K(0)$ is the initial capital stock. Equation (2) describes the production frontier based on Benhabib and Nishimura (1998), where $\hat{C}$ and $\hat{I}$ is output level of consumption and investment goods.[3] The private factor shares of labor and capital in sector $i = I, C$ are measured by a_i and b_i and the externalities associated with labor and capital are given by $\overline{L_i}$ and $\overline{K_i}$. In both sectors, production exhibits constant returns to scale from the social perspective, i.e. $a_i + \alpha_i + b_i + \beta_i = 1$, but decreasing returns to scale from the private perspective, i.e. $a_i + b_i < 1$. Equation (3) describes the resource constraints for capital and labor, both of which are perfectly mobile across sectors. Equation (4) is capital accumulation process where $\delta \in (0,1)$ is the capital depreciation allowance. The current value Hamiltonian is

$$
\begin{aligned}
\mathrm{H} = \frac{C^{1-\eta}}{1-\eta} &+ \lambda(\hat{I} + p\hat{C} - \delta K - pC) + P_I(L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} - \hat{I}) \\
&+ P_C(L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} - \hat{C}) + \overline{r}(K - K_I - K_C) + \overline{w}(L - L_I - L_C)
\end{aligned}
$$

where P_I, P_C, $\overline{r}$ and $\overline{w}$ are the Lagrange multipliers representing the shadow price of investment, consumption, capital and labor respectively. The first order conditions with respect to $\hat{I}$ and $\hat{C}$ are

$$\lambda = P_I$$

$$\lambda p = P_C$$

Combining the above, one has $p = P_C/P_I$. Defining $r = \overline{r}/P_I$ and $w = \overline{w}/P_I$, we can express the remaining necessary conditions as

$$C^{-\eta} = \lambda p \tag{5}$$

$$w = a_I L_I^{a_I - 1} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p a_C L_C^{a_C - 1} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{6}$$

$$r = b_I L_I^{a_I} K_I^{b_I - 1} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p b_C L_C^{a_C} K_C^{b_C - 1} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{7}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r) \tag{8}$$

$$\lim_{t \to \infty} K(t)\lambda(t)e^{-\rho t} = 0 \tag{9}$$

Equation (5) expresses the equality between the marginal utility and the marginal cost of consumption while (8) represents the intertemporal arbitrage condition. Equations (6) and (7) state the equalization of the marginal revenue product of labor and of capital across sectors. Equation (9) is the transversality condition. Note that (6) and (7) consist of four equations in four unknowns in L_I, L_C, K_I and K_C. They are implicitly solved in terms of w and r, which are in turn determined after p is obtained. The relationship between total income from the private and the social perspectives can be expressed as

$$wL + rK + \Pi = W_I L_I + R_I K_I + W_C L_C + R_C K_C \left(= \hat{I} + p\hat{C}\right) \tag{10}$$

[3] Since this is a two-country model, the distinction between the actual consumption and investment levels with the actual output levels has to be made because only under autarky can we conclude that market clearing implies $C = \hat{C}$ and $I = \hat{I}$.

where Π is the firm's profit due to private decreasing returns to scale, W_i and R_i are the social wage and rental income in sector i respectively. Using the fact that production functions exhibit constant returns to scale, L_i and K_i can be expressed as functions of W_i and R_i. This is done by setting $L_i = \overline{L_i}$, $K_i = \overline{K_i}$, $a_i + \alpha_i = \theta_i$ and $b_i + \beta_i = 1 - \theta_i$ in (2) and observing that

$$W_i = p_i \theta_i L_i^{\theta_i - 1} K_i^{1-\theta_i} \tag{11}$$

$$R_i = p_i \left(1 - \theta_i\right) L_i^{\theta_i} K_i^{-\theta_i} \tag{12}$$

where

$$p_i = \begin{cases} 1 & \text{if } i = I, \\ p & \text{if } i = C. \end{cases}$$

Note that W_i and R_i may differ across sectors, since factor allocation across sectors only depends on private returns, which are equalized for each input in equilibrium. Equations (11) and (12) can be equivalently expressed as

$$W_i l_i = p_i \theta_i \tag{13}$$

$$R_i k_i = p_i \left(1 - \theta_i\right) \tag{14}$$

where l_i and k_i are the unit labor and capital requirement respectively. Using (13) and (14), the two social zero profit conditions in the goods markets are

$$1 = W_I l_I + R_I k_I \tag{15}$$

$$p = W_C l_C + R_C k_C \tag{16}$$

Due to social constant returns to scale, the unit input requirements are functions of social income alone. Equations (11) and (12), combined with (6) and (7) yield two useful relationships:

$$W_i = \frac{\theta_i}{a_i} w \tag{17}$$

$$R_i = \frac{1 - \theta_i}{b_i} r \tag{18}$$

By substituting the social income levels from (17) and (18) into (15) and (16), and log-differentiating the social zero profit functions with respect to p, we obtain the Stolper-Samuelson conditions as

$$\frac{p r'(p)}{r(p)} = \frac{\theta_I}{\theta_I - \theta_C} \tag{19}$$

$$\frac{p w'(p)}{w(p)} = \frac{1 - \theta_I}{\theta_C - \theta_I} \tag{20}$$

where $\theta_C - \theta_I$ measures the social factor intensity ranking. $\theta_C - \theta_I > 0$ holds if labor is socially more intensive in the consumption sector, and the inequality reverses if the converse is true. It turns out that the Stolper-Samuelson conditions, which express the price elasticities of wage and rental income, are the same even when production externalities are absent.

Applying Shephard's Lemma to (15) and (16), the factor clearing conditions can be expressed as

$$\begin{pmatrix} l_I & l_C \\ k_I & k_C \end{pmatrix} \begin{pmatrix} \hat{I} \\ \hat{C} \end{pmatrix} = \begin{pmatrix} L \\ K \end{pmatrix} \tag{21}$$

The system of equations (21) contains two unknowns $\hat{I}$ and $\hat{C}$, which can be solved using (13), (14), (17) and (18) to obtain

$$\hat{I} = \frac{b_C w(p) L - a_C r(p) K}{\Delta} \tag{22}$$

$$\hat{C} = \frac{a_I r(p) K - b_I w(p) L}{p\Delta} \tag{23}$$

where $\Delta = a_I b_C - a_C b_I$ measures the private factor intensity ranking. In particular, $\Delta > 0$ can be obtained with $a_I > a_C$, where labor is privately more intensive in the investment sector, together with $b_C > b_I$, where capital is privately more intensive in the consumption sector. Substituting $\hat{I}$ and $\hat{C}$ from (22) and (23) into (4), we have

$$\dot{K} = \frac{a_I r(p) K - b_I w(p) L}{\Delta} + \frac{b_C w(p) L - a_C r(p) K}{\Delta} - \delta K - pC \tag{24}$$

Equations (8) and (24) govern the law of motion for the home country.

The foreign country differs from the home country through the absence of sector-specific externalities. Distinguishing the foreign country's variables by the asterisk, the production functions are

$$\hat{I}^* = L_I^{*\theta_I} K_I^{*1-\theta_I} \quad \text{and} \quad \hat{C}^* = L_C^{*\theta_C} K_C^{*1-\theta_C} \tag{25}$$

Assume that the size of the labor force is the same in both countries. The equations governing the evolution of the foreign country are

$$\dot{K}^* = w^*(p) L + r^*(p) K^* - \delta^* K^* - pC^* \tag{26}$$

$$\dot{\lambda}^* = \lambda^* \left(\rho^* + \delta^* - r^*(p)\right) \tag{27}$$

where $p = p^*$ holds with trade. Equations (26) and (27), together with (8) and (24) determine the dynamics of the two-country world economy. Since social technologies are the same as the home country's, the Stolper-Samuelson conditions will be

$$\frac{p r^{*\prime}(p)}{r^*(p)} = \frac{\theta_I}{\theta_I - \theta_C} \tag{28}$$

$$\frac{pw^{*\prime}(p)}{w^*(p)} = \frac{1-\theta_I}{\theta_C - \theta_I} \tag{29}$$

In equilibrium, the trade balance for consumption good is given by

$$(\lambda p)^{-1/\eta} + (\lambda^* p)^{-1/\eta} = \frac{a_I r(p)K - b_I w(p)L}{p\Delta} + \frac{\theta_I r^*(p)K^* - (1-\theta_I)w^*(p)L}{p(\theta_I - \theta_C)} \tag{30}$$

To simplify the analysis, we reduce the system by one dimension. The following lemma and assumption are useful for this purpose.

Lemma 1. *There is a constant $\xi > 1$, determined from $r^*(0) = \xi r(0)$, that solves $r^*(t) = \xi r(t)$ for $t \in [0, \infty)$.*

Proof. See Appendix 1. □

Assumption 1. $\frac{\rho^*+\delta^*}{\xi} = \rho + \delta$.

The condition implied by this assumption is time invariant since ξ is a constant. Given the above, we have

Lemma 2. *Under Lemma 1 and Assumption 1, $\lambda^* = m\lambda^\xi$ holds for $t \in [0, \infty)$.*

Proof. By Lemma 1, (27) is equivalent to

$$\frac{\dot{\lambda}^*}{\lambda^*}\frac{1}{\xi} = \frac{\rho^* + \delta^*}{\xi} - r(p)$$

which together with Assumption 1 can be written as

$$\frac{\dot{\lambda}^*}{\lambda^*} = \xi \frac{\dot{\lambda}}{\lambda}$$

This expression is integrated to obtain $\lambda^* = m\lambda^\xi$, where $m > 0$ is a constant. □

Using Lemma 2, the model's behavior is described by the following system of three equations

$$\dot{K} = \frac{a_I r(p)K - b_I w(p)L}{\Delta} + \frac{b_C w(p)L - a_C r(p)K}{\Delta} - \lambda^{-1/\eta} p^{1-1/\eta} - \delta K \tag{31}$$

$$\dot{K}^* = w^*(p)L + r^*(p)K^* - (m\lambda^\xi)^{-1/\eta} p^{1-1/\eta} - \delta^* K^* \tag{32}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r(p)) \tag{33}$$

The steady state price level is obtained by setting $\dot{\lambda} = 0$ in (33). Totally differentiating the trade balance equation for consumption yields

$$\frac{dp}{dK} = -\frac{\Sigma a_I(\rho+\delta)}{\Delta\left((\lambda p)^{-1/\eta} + (m\lambda^\xi p)^{-1/\eta}\right)} \tag{34}$$

$$\frac{dp}{dK^*} = -\frac{\Sigma\theta_I(\rho^* + \delta^*)}{(\theta_I - \theta_C)\left((\lambda p)^{-1/\eta} + (m\lambda^\xi p)^{-1/\eta}\right)} \tag{35}$$

$$\frac{dp}{d\lambda} = -\frac{p\Sigma\left((\lambda p)^{-1/\eta} + \xi(m\lambda^\xi p)^{-1/\eta}\right)}{\eta\lambda\left((\lambda p)^{-1/\eta} + (m\lambda^\xi p)^{-1/\eta}\right)} \tag{36}$$

where

$$\frac{1}{\Sigma} = \frac{1}{\eta} - 1 + \frac{\frac{\theta_I r^{*\prime}(p)K^* - (1-\theta_I)w^{*\prime}(p)L}{\theta_I - \theta_C} + \frac{a_I r'(p)K - b_I w'(p)L}{\Delta}}{(\lambda p)^{-1/\eta} + (m\lambda^\xi p)^{-1/\eta}} \tag{37}$$

Define $\gamma = \left(\frac{(a_I - a_C)r(p) - \delta\Delta}{\Delta}\right)$. Using (34), (35), (36) and (37), the linearization of (31), (32), and (33) yields Jacobian matrix J with determinant

$$Det(J) = \frac{pr'(p)}{\eta}\left(-\rho^*\gamma\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + \rho^*(\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi\gamma(m\lambda^\xi p)^{-1/\eta}\frac{dp}{dK^*}\right) \tag{38}$$

Assumption 2. *$\rho^* \geq \gamma$.*

Assumption 2 is imposed for simplification purpose alone and is very easily satisfied under reasonable parameterization. In addition, we consider $w^{*\prime}(p)L + r^{*\prime}(p)K^* > 0$, which says that national income of the foreign country increases with the relative price of consumption. Since consumption good is socially labor intensive, a country with a sufficiently large labor-capital ratio will fulfill this requirement.

Now consider the following cases. First, suppose $\Delta < 0$ and $\theta_I - \theta_C < 0$. In this case, factor intensity rankings from the private and social perspectives are the same.

Definition. *The equilibrium path is a sequence $\{p(t), w(t), w^*(t), r(t), r^*(t)\}_{t=0}^\infty$ and $\{K(t), K^*(t), \lambda(t)\}_{t=0}^\infty$ such that for each t, the sequence i) solves the representative agent's problem, ii) satisfies (31), (32) and (33) given the initial stock of capital and the transversalilty condition for each country, and iii) clears all factor markets and trade balances.*

Proposition 1. *Under Assumption 1, $\Delta < 0$ and $\theta_I - \theta_C < 0$, the equilibrium path is unique.*

Proof. The model's behavior can be analyzed by straightforward application of the Routh Theorem.[4] Since $\Delta < 0$ and $\theta_I - \theta_C < 0$, we know that $\Sigma > 0$, which implies that both dp/dK and dp/dK^* are positive and $dp/d\lambda$ is negative. Moreover, $\theta_I - \theta_C < 0$ implies that $r'(p) < 0$ and therefore $Det(J) < 0$. Since $Trace(J)$ contains both positive and negative terms, we consider two cases:

[4]The Routh Theorem states that the number of changes in signs in the following scheme

$$-1 \qquad Trace(J) \qquad -F + \frac{Det(J)}{Trace(J)} \qquad Det(J)$$

indicates the number of eigenvalues with positive parts, where F is the sum of the minor matrices $c_{11}c_{22} - c_{21}c_{12}$, $c_{11}c_{33} - c_{31}c_{13}$, $c_{22}c_{33} - c_{23}c_{32}$.

1. $Trace(J) > 0$. The signs must change twice, regardless of the sign of $-F + \frac{Det(J)}{Trace(J)}$. Hence, there are two eigenvalues with positive real part and one with negative real part. Since λ is the only non-predetermined variable, the steady state is saddle-path stable.

2. $Trace(J) < 0$. Appendix 2 demonstrates that $-F + \frac{Det(J)}{Trace(J)}$ is always positive if $Trace(J) < 0$. Once again, the signs must change twice. Hence, the steady state is always saddle-path stable. □

Next, consider $\Delta > 0$, $\theta_I - \theta_C < 0$ and $\Sigma > 0$. In this case, private and social factor intensity rankings differ.

Proposition 2. *Under Assumption 1, $\Delta > 0$, $\theta_I - \theta_C < 0$ and $\Sigma > 0$, the equilibrium path is unique if*

$$\frac{C^*}{C} \in \left(\frac{\rho^*(a_I(\rho+\delta) - \gamma\Delta)(\theta_C - \theta_I)}{\xi\Delta\gamma(\rho^*\theta_C + \delta^*\theta_I)}, \infty \right) \tag{39}$$

Proof. With $r'(p) < 0$, the determinant is negative as long as

$$-\rho^*\gamma\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + \rho^*(\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi\gamma(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*} > 0 \tag{40}$$

If the trace is positive, the steady state is saddle-path stable. Otherwise, saddle-path stability arises for $-F + \frac{Det(J)}{Trace(J)} > 0$. This is satisfied as long as (40) holds (see Appendix 3). Using (34), (35), (36), and the fact that $C = (\lambda p)^{-1/\eta}$ and $C^* = (m\lambda^{\xi}p)^{-1/\eta}$, we can express (40) as (39). □

We state without proof that under autarky, indeterminacy arises in the home country for $\Delta > 0$, $\theta_I - \theta_C < 0$ and $\Sigma > 0$ while the equilibrium path is always unique for the foreign country (available upon request.) Note that if $\Delta < 0$ and $\theta_I - \theta_C < 0$, then there is no factor intensity reversal and (39) always holds. This reduces to Proposition 1, that is, the equilibrium path is always unique when there is no factor intensity reversal. The outcome is explained by the fact that the steady state in both country is saddle-path stable under autarky to begin with. Hence, one can view Proposition 2 as a generalization of Proposition 1.

More importantly, by relaxing the assumption of identical social technology in Nishimura and Shimomura (2002), one in which sector-specific externalities are removed from one country, indeterminacy is eliminated whenever (39) is satisfied. In other words, Proposition 2 asserts that even though indeterminacy may arise under autarky, determinacy may be effected through trade. If the dynamic properties under autarky are preserved under trade, then examining the closed economy is appropriate. Otherwise, while indeterminacy may emerge from analyzing a closed economy framework, determinacy may be a possible outcome with trade.

To determine the restrictiveness of the sufficient condition, consider the benchmark model:

[Insert Table 1 here]

Consistent with the empirical findings, we have assumed very small numerical quantities of externalities. In addition, we impose $a_I > a_C$ with $b_I < b_C$ so that $\Delta > 0$, otherwise indeterminacy may not arise. Hence, the benchmark parameters imply that $\theta_I - \theta_C = -0.0001$ and $\Delta = 0.01247$. The lower bound for sufficiency is larger the closer ξ is to one since $\xi > 1$. Therefore, we set $\xi = 1$, the limiting case, so that if C^*/C satisfies the sufficient condition for $\xi = 1$, it will satisfy the sufficient condition for any $\xi > 1$. Finally, we let $\rho = 0.05$, $\delta = 0.05$, $\rho^* = 0.05$ and $\delta^* = 0.05$. Since $\gamma = 0.03019$, ρ^* satisfies Assumption 2. Given these values, the sufficient condition is met as long as C^*/C is greater than 0.01307. We have experimented with different values ρ, ρ^*, δ, δ^* but the lower bound for sufficiency remains small.[5] For example, if sector-specific externalities are absent in the U.S. major trading partners, say Canada and Japan, then the private consumption ratios (in 1997) of 0.05738 with Canada and 0.50633 with Japan indicate that indeterminacy may not emerge in the U.S. after all even though it appears to be so when viewed from a closed economy perspective.[6]

3 Conclusion

The motivation of this paper is to highlight the possibility that indeterminacy may arise under autarky while determinacy is an outcome given trade. However, with the simplifying assumptions, our finding should not be viewed as suggesting that past results based on analyzing the closed economy are not robust to open economy extensions. Rather, our purpose is to point out that such non-robustness may be present. A limitation of this paper has been the technical involvement that makes it difficult to obtain clear economic intuition on our finding, even though the model presented here is based upon a well-known dynamic Heckscher-Ohlin framework of Nishimura and Shimomura (2002) with the injection of production externalities only in one country. Deeper investigation is necessary for the better understanding of why and when determinacy, or indeterminacy, dominates when trade takes place between two countries with one experiencing determinacy and the other indeterminacy under autarky. For instance, it would be interesting to carry out a similar exercise on previous research by extending these studies to an open economy Ricardian setting, so that one can inspect the mechanism through which the dynamic properties are transmitted across the countries and examine the role, if any, of the current account and the terms of trade more closely. Should sunspots persist in the open economy, then whether there are useful policies in eliminating sunspots and if so how policies should be conducted, in a coordinated or non-coordinated fashion, are interesting questions on their own. Moreover, since this paper only considers trade linkages, it

[5] From the benchmark, we vary in a pairwise manner i) ρ and ρ^*, ii) ρ and δ^*, and iii) ρ^* and δ from 0.01 to 0.1 while maintaining $\xi = 1$. To be more precise, for example for i), we keep $\delta = \delta^* = 0.05$ and vary $\rho = \rho^*$ between 0.01 to 0.1. The largest lower bound we find is 0.03916, which is obtained for $\rho = \delta^* = 0.05$ and $\rho^* = \delta = 0.1$. We have also tried different plausible combinations of the parameters and the finding of a small lower bound generally remains.

[6] Based on our calculation from the OECD Business Sector Database.

would also be worthwhile to consider whether financial linkages could act as a channel for transmitting endogenous fluctuations across countries. On this note, we hope to encourage further research examining the robustness of indeterminacy to open economy extensions, and more importantly in general, the transmission of equilibrium dynamics through international macroeconomic interdependence.

Appendix 1

The time argument is introduced for expositional sake. Given $K(0)$ and $K^*(0)$, the initial price $p(0)$ is determined from (30) after $\lambda(0)$ and $\lambda^*(0)$ are chosen. With $p(0)$, $r(0) = r(p(0))$ and $r^*(0) = r^*(p(0))$ are obtained. Hence, there is a constant $\xi > 1$ that solves

$$r^*(0) = \xi r(0)$$

The evolution of $p(t)$ depends on $K(t)$, $K^*(t)$, $\lambda(t)$ and $\lambda^*(t)$. In turn, the growth paths of $r(t)$ and $r^*(t)$ depend on $p(t)$. To determine the relationship between the instantaneous growth rates of $r(t)$ and $p(t)$, we approximate (19) by

$$\frac{dr(t)}{dt}\frac{1}{r(t)} = \left(\frac{\theta_I}{\theta_I - \theta_C}\right)\frac{dp(t)}{dt}\frac{1}{p(t)}$$

which is redefined as

$$g_{r(t)} = \left(\frac{\theta_I}{\theta_I - \theta_C}\right) g_{p(t)} \qquad \text{for} \qquad t \in [0, \infty) \tag{A1}$$

where $g_{r(t)}$ and $g_{p(t)}$ are the instantaneous growth rate of $r(t)$ and $p(t)$. Since $p(t)r'(p(t))/r(p(t)) = p(t)r^{*\prime}(p(t))/r^*(p(t))$, it is also true that

$$g_{r^*(t)} = \left(\frac{\theta_I}{\theta_I - \theta_C}\right) g_{p(t)} \qquad \text{for} \qquad t \in [0, \infty) \tag{A2}$$

where $g_{r^*(t)}$ is the instantaneous growth rate of $r^*(t)$. At $t = 0$, the instantaneous growth of K, K^*, λ and λ^* along the equilibrium path causes relative price to grow at a rate $g_{p(0)}$. Together with (A1) and (A2), $g_{p(0)}$ pins down $g_{r(0)}$ and $g_{r^*(0)}$. For an infinitesimally small distance, dt, from $t = 0$, the interest rates are

$$\begin{aligned} r(dt) =& (1 + g_{r(0)})r(0) \\ r^*(dt) =& (1 + g_{r^*(0)})r^*(0) \end{aligned}$$

Since $g_{r(0)} = g_{r^*(0)}$ and $r^*(0) = \xi r(0)$, we have $r^*(dt) = \xi r(dt)$. Extend the argument from period $t = 0$ to $t = \bar{t}$ and define $\gamma_{r(\bar{t})}$ and $\gamma_{r^*(\bar{t})}$ as the average growth rate of $r(t)$ and $r^*(t)$ from $t = 0$ to

$t = \bar{t}$. One has

$$r(\bar{t}) = e^{\gamma_{r(\bar{t})}\bar{t}} r(0)$$
$$r^*(\bar{t}) = e^{\gamma_{r^*(\bar{t})}\bar{t}} r^*(0)$$

Since $g_{r(t)} = g_{r^*(t)}$ holds for $t \in [0, \infty)$, therefore $\gamma_{r(\bar{t})} = \gamma_{r^*(\bar{t})}$ and hence $r(\bar{t}) = \xi r^*(\bar{t})$ holds for $\bar{t} \in [0, \infty)$. Since $\bar{t}$ is arbitrary, set $\bar{t} = t$ to obtain $r(t) = \xi r^*(t)$ for $t \in [0, \infty)$. $\square$

Appendix 2

The Jacobian matrix is

$$J = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

where

$$c_{11} = \Lambda \frac{dp}{dK} + \frac{(a_I - a_C) r(p) - \delta\Delta}{\Delta} \qquad c_{12} = \Lambda \frac{dp}{dK^*} \qquad c_{13} = \Lambda \frac{dp}{d\lambda} + \frac{p(\lambda p)^{-1/\eta}}{\eta\lambda}$$
$$c_{21} = \Gamma \frac{dp}{dK} \qquad c_{22} = \rho^* + \Gamma \frac{dp}{dK^*} \qquad c_{23} = \Gamma \frac{dp}{d\lambda} + \frac{\xi p(m\lambda^{\xi} p)^{-1/\eta}}{\eta\lambda}$$
$$c_{31} = -\lambda r'(p) \frac{dp}{dK} \qquad c_{32} = -\lambda r'(p) \frac{dp}{dK^*} \qquad c_{33} = -\lambda r'(p) \frac{dp}{d\lambda}$$

and

$$\Lambda = \frac{a_I r'(p) K - b_I w'(p) L}{\Delta} + \frac{b_C w'(p) L - a_C r'(p) K}{\Delta} + \frac{(1-\eta)(\lambda p)^{-1/\eta}}{\eta}$$
$$\Gamma = w^{*\prime}(p) L + r^{*\prime}(p) K^* + \frac{(1-\eta)(m\lambda^{\xi} p)^{-1/\eta}}{\eta}$$

The proof that $-F + \frac{Det(J)}{Trace(J)} > 0$ given $Trace(J) < 0$ is as follows. First, write $-F + \frac{Det(J)}{Trace(J)} > 0$ as $-\frac{1}{Trace(J)}(F\,Trace(J) - Det(J))$. Since $-\frac{1}{Trace(J)} > 0$, $-F + \frac{Det(J)}{Trace(J)} > 0$ if and only if $F\,Trace(J) - Det(J) > 0$. We define

$$X = c_{11}c_{22} - c_{21}c_{12} = \rho^*\gamma + \rho^*\Lambda \frac{dp}{dK} + \gamma\Gamma \frac{dp}{dK^*}$$
$$Y = c_{22}c_{33} - c_{23}c_{32} = -\rho^*\lambda r'(p) \frac{dp}{d\lambda} + r'(p) \frac{\xi p}{\eta} (m\lambda^{\xi} p)^{-1/\eta} \frac{dp}{dK^*}$$
$$Z = c_{11}c_{33} - c_{31}c_{13} = -\gamma\lambda r'(p) \frac{dp}{d\lambda} + r'(p) \frac{p}{\eta} (\lambda p)^{-1/\eta} \frac{dp}{dK}$$

where $F = X + Y + Z$. Write $F\,Trace(J)$ as

$$
\begin{aligned}
&(X+Y+Z)\,Trace(J)\\
&= Trace(J)\,X + \left(\rho^* + \Lambda\frac{dp}{\partial K} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y + \gamma Y\\
&- Trace(J)\,\gamma\lambda r'(p)\frac{dp}{d\lambda} + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)\\
&+ \gamma r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK} \qquad \text{(A3)}
\end{aligned}
$$

The second line is derived by expanding $Trace(J)$ in $Trace(J)\,Y$, the third and last line are derived by expanding both $Trace(J)$ and Z in $Trace(J)\,Z$. After rearranging, we obtain

$$
\begin{aligned}
&(X+Y+Z)\,Trace(J) - \gamma\left(Y + r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)\\
&= Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda}) + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y\\
&+ r'(p)\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)
\end{aligned}
$$

Now, using the expression for X,

$$
\begin{aligned}
&X - \gamma\lambda r'(p)\frac{dp}{d\lambda}\\
&= \rho^*\gamma + \rho^*\Lambda\frac{dp}{dK} + \gamma\Gamma\frac{dp}{dK^*} - \rho^* r'(p)\frac{dp}{d\lambda} + (\rho^* - \gamma)\lambda r'(p)\frac{dp}{d\lambda} \qquad \text{(A4)}
\end{aligned}
$$

If $\rho^* - \gamma$ is small and ρ and δ are not too large, the last term in (A4) is small. If $\rho^* = \gamma$, this term drops out. Using Assumption 2, we approximate (A4) as

$$
\begin{aligned}
&X - \gamma\lambda r'(p)\frac{dp}{d\lambda}\\
&\approx \rho^*\gamma + \rho^*\Lambda\frac{dp}{dK} + \gamma\Gamma\frac{dp}{dK^*} - \rho^* r'(p)\frac{dp}{d\lambda}\\
&< \rho^*\left(\gamma + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - r'(p)\frac{dp}{d\lambda}\right) < 0 \qquad \text{(A5)}
\end{aligned}
$$

since $Trace(J) < 0$ implies $\left(\gamma + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{d}{d\lambda}\right) < 0$. Hence, $Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda}) > 0$. Next, since $Y < 0$, we have

$$
\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y > 0
$$

Furthermore, $dp/dK > 0$ and $r'(p) < 0$ imply

$$r'(p)\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right) > 0$$

Therefore, $F\,Trace(J) - \gamma\left(Y + r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right) > 0$. It can be easily verified that $-Det(J) \geq -\gamma\left(Y + r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)$. Hence $F\,Trace(J) - Det(J) > 0$. $\square$

Appendix 3

The proof that $-F + \frac{Det(J)}{Trace(J)} > 0$ given $\left(-\rho^*\gamma\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + \rho^*(\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi\gamma(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*}\right) > 0$ and $Trace(J) < 0$ is as follows. Expand Y in (A3) and add and subtract $\rho^* r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}$ from the RHS to obtain

$$\begin{aligned}
&F\,Trace(J) - Det(J)\\
&= Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda})\\
&+ \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{pr'(p)}{\eta}\right)\\
&\times\left(-\rho^*\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + (\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*}\right)\\
&+ (\gamma - \rho^*)r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}
\end{aligned} \tag{A6}$$

Again, if $\gamma - \rho^*$ is small and ρ and δ are not too large, the last term in (A6) is small. This term is zero if $\gamma = \rho^*$. Hence, we approximate (A6) as

$$\begin{aligned}
&F\,Trace(J) - Det(J)\\
&\approx Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda})\\
&+ \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{pr'(p)}{\eta}\right)\\
&\times\left(-\rho^*\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + (\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*}\right)
\end{aligned}$$

Now, $Trace(J) < 0$ and $r'(p) < 0$ implies $\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{pr'(p)}{\eta}\right) > 0$. It is easy to verify that (40) implies $\left(-\rho^*\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + (\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*}\right) > 0$. Finally, $Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda}) > 0$ as before. Therefore, $F\,Trace(J) - Det(J) > 0$. $\square$

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Table 1: Benchmark specification of the model economy

$a_I = 0.3$	$\alpha_I = 0.001$	$b_I = 0.68$	$\beta_I = 0.019$	$\rho = 0.05$	$\delta = 0.05$
$a_C = 0.29$	$\alpha_C = 0.0111$	$b_C = 0.6989$	$\beta_C = 0$	$\rho^* = 0.05$	$\delta^* = 0.05$