Work quality and optimal pay structure: Piece vs. hourly rates in employee remuneration

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Abstract

This paper constructs a model investigating the impact of pay structure on the quantity and quality of work and its welfare implications. We prove that a change in pay structure alone could increase the utility of employers given the employees' utility level, and hence, by appropriate income distribution, could increase social welfare.

Keywords: Optimal pay structure; Work quality; Employee remuneration

JEL classification: J33

1. Introduction

An employer of translators complained, "If you pay according to piece-rate (e.g. $x per thousand words), the quality of the work is low, and if you pay hourly rates, the quality is higher but the work is slow." It thus occurred to us that a better structure of pay is to adopt a combination of both piece-rates and hourly rates. We were surprised to find no formal analysis of this obviously important issue after a search of the literature. The answer to this puzzle came readily as we analysed the problem ourselves: it is mathematically too complicated to get a neat analytical result.

The complication arises not just from the fact that the employer's optimisation problem (choice of the optimal combination of piece and hourly rates) has to be subject to the employee's optimisation behaviour. Even the employee's problem itself is already too complicated to solve analytically. This complication arises from the fact that not only should we let income and leisure enter the employee's utility function, we must also include an index for the intensity of work as well as an output index (which is itself a function of quality and quantity). The reasons are clear: if we do not let employees choose the intensity of work, they...

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will not be able to vary the quality/quantity mixture depending on the pay structure. Secondly, if employees do not care about the output index, they will simply choose zero quality under piece-rates and zero quantity under hourly rates. Thus, from the nature of our problem, we must let employees choose the intensity of work. The employee’s interest in the output index could be explained by concern for reputation, need to maintain self-esteem, or fear of dismissal if performance falls below the employer’s minimum standards.

Given the necessity to include both an intensity and an output index into the utility function (on top of income and leisure), the consumer optimisation problem itself becomes too complicated (for us anyway) to solve analytically. (We can derive the first-order conditions and the comparative-statics and cannot sign the latter; see Appendix 1.) We then go from a general utility function to a Cobb–Douglas one. Thanks to the modern software technology, we are able to solve the optimisation problem symbolically, though the comparative-statics still prove too difficult to sign for general (algebraic) values of the parameters. Next, we move to adopt specific numerical values and calculate how the relevant variables change with changes in the pay structure. We then obtain the expected results: a higher piece-rate component decreases the quality of work (Proposition 1). For some reasonable index of output as a concave function of quantity and quality, the optimal structure of pay is a combination of piece and hourly rates (Proposition 2). Holding employee’s utility unchanged, net output can be increased by using a mixed pay structure. Conversely, holding net output unchanged, employee’s utility can be increased by using a mixed pay structure in comparison to either pure piece-rates or pure hourly rates. The second result is more important than the first in terms of practical significance as the second proposition may mean that substantial productivity gains could be made by merely changing our compensation practices. This raises the question why the mixed pay structure has not been observed more often. Can administrative simplicity be a sufficient explanation or is some inadequate understanding of the optimal pay structure also contributing? If so, can our demonstration help to generate more mixed structures? The answers to these questions are however beyond the scope of this paper.

While our demonstration depends on a specific utility function (Cobb–Douglas), we believe that the result can be generalised to other reasonable utility functions and to a more general utility function. Even before such generalisation, since the Cobb–Douglas is a sort of bench-mark case, our results should prove to be sufficiently interesting.

Our problem is similar to, but different from, the principal–agent problem. The latter arises when the agent (e.g. worker, farmer) does a job for the principal (employer, landlord) where the quality of the work done is difficult to distinguish because of some random factors (e.g. weather). The principal–agent problem has been fairly extensively studied [see, eg. Allen (1988), Cheung (1969), Ross (1973), Shavell (1979), Stiglitz (1974)]. Most of this literature concentrates on the relationship between risk aversion and the optimal degree of “sharecropping”. Our problem also arises from the difficulty of specifying quality. However, there is no uncertainty on our model and the issue of risk aversion does not arise. We ignore the possibility of sharecropping or payment in accordance to output and concentrate on the choice between piece and hourly rates. Payment in accordance to output may be impracticable where the output index (partly dependent on quality) may be difficult to define, such as in the case of translation mentioned above.
2. Analysis

2.1. Employee’s optimisation problem

The employee’s utility function depends positively on income, leisure, the output index, and negatively on the intensity of work.

\[ U = U(Y, L, I, \Phi) , \quad U_Y > 0, U_L < 0, U_I < 0, U_{\Phi} > 0 . \]  

(1)

where \( U \) = utility, \( Y \) = income, \( L \) = labour (hours of work), \( I \) = intensity of work, \( \Phi \) = output index.

For the specific Cobb–Douglas function, we have

\[ U = Y^y (L - L)^{(L - I)'} \Phi^\phi \]

(1')

where \( y, l, i, \phi \) are all positive constants, and \( \bar{L} \) and \( \bar{I} \) are some absolute upper limits on working hours and intensity.

The output index \( \Phi \) is taken as a Cobb–Douglas function of quantity \((X)\) and quality \((Q)\) of production.

\[ \Phi = X^\gamma Q^\delta \]

(2)

It might be believed that perhaps only the quality \( Q \) should enter the employee’s utility function, since quantity \( X \) should only affect utility indirectly through its effect on income \( Y \). This belief is questionable. First, letting only \( Q \) enter the utility function may work if we are dealing only with a pure piece-rate system. However, under the pure hourly rate system, \( X \) will fall to zero if it does not enter the utility function. Since we are dealing with a general system with pure hourly rate as a special case, we must let \( X \) enter the utility function directly as well. Secondly, the productivity of a given quality to the employer depends on \( X \). If an employee achieves very high quality but always produces a negligible quantity, it is not of much use to the employer. Thus, reputation should depend on the output index instead of just the quality.

Income earned is given by,

\[ Y = \alpha w X + (1 - \alpha) s L \]

(3)

where \( X \) is the number of units produced, \( w \) is the (pure) piece wage-rate, \( s \) the (pure) hourly salary, and \( \alpha \) the measure of piece-pay component in the mixed structure.

We adopt a simple trade-off between quantity \( X \) and quality \( Q \) and a simple complementarity relation between hours and intensity of work:

\[ X + Q = LI \]

(4)

While this is not a general function, the essential trade-off has been captured.

After substituting \( X \) from (4) into (3) and substituting \( Y \) from the resulting equation and \( \Phi \)
from (2) into (1'), we let the employee maximise (1') with respect to $I, L,$ and $Q,$ taking $w, s,$ and $\alpha$ as given. The following first-order conditions can be derived:

\[
\frac{yawL}{Y} + \frac{x\phi L}{X} = \frac{i}{I - I} \tag{5}
\]

\[
y[awl + (1 - \alpha)s] + \frac{x\phi l}{X} = \frac{l}{L - L} \tag{6}
\]

\[
\frac{q\phi}{Q} = \frac{x\phi}{X} + \frac{yaw}{Y} \tag{7}
\]

Equations (2)-(7) determine the utility-maximising solutions for $I, Q, L, X, Y, \Phi,$ for given $y, l, i, \phi, x, q, w, s,$ and $\alpha.$

\[
Q = Q(y, l, i, q, \phi, x, w, s, \alpha) ; \tag{8}
\]

\[
L = L(y, l, i, q, \phi, x, w, s, \alpha) ;
\]

\[
I = I(y, l, i, q, \phi, x, w, s, \alpha) ;
\]

\[
Y = Y(y, l, i, q, \phi, x, w, s, \alpha) ;
\]

substituting (8) into (1'), we have employee's indirect utility function $U^*$,

\[
U^* = U^*(y, l, i, q, \phi, x, w, s, \alpha) \tag{9}
\]

We intuitively believe that an increase in $\alpha$ (piece-rate component) should decrease $Q$ (quality); the comparative-static analysis\(^2\) and a numerical illustration confirm this intuition.

\[
dQ/\alpha = f(y, l, i, q, \phi, x, w, s, \alpha) < 0 . \tag{10}
\]

We apply a numerical example to illustrate that the pay structure might influence the quality of work, and an increase in piece-rate component ($\alpha$) should decrease the quality of work (see Table 1).

We may state the above result as,

**Proposition 1**: An increase in the piece-rate component in the pay structure decreases the quality of work.

While the result is formally established only for the Cobb–Douglas case, we believe that it is generalisable.

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1 To save space, we only list the general solution of this nonlinear equation system in the text. The explicit solutions are provided in Appendix 2.

2 To get the comparative static result, we apply two different approaches. The direct approach is to differentiate the variables solved in Appendix 2; and the indirect approach is to totally differentiate the first-order condition (5)-(7), and (3) and solve for the results in elasticity form. The second approach is reported in Appendix 1.
Table 1

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$l$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.514</td>
<td>18.217</td>
</tr>
<tr>
<td>0.2</td>
<td>0.532</td>
<td>17.532</td>
</tr>
<tr>
<td>0.3</td>
<td>0.556</td>
<td>16.667</td>
</tr>
<tr>
<td>0.4</td>
<td>0.583</td>
<td>15.625</td>
</tr>
<tr>
<td>0.5</td>
<td>0.614</td>
<td>14.462</td>
</tr>
<tr>
<td>0.6</td>
<td>0.646</td>
<td>13.269</td>
</tr>
<tr>
<td>0.7</td>
<td>0.677</td>
<td>12.13</td>
</tr>
<tr>
<td>0.8</td>
<td>0.704</td>
<td>11.095</td>
</tr>
<tr>
<td>0.9</td>
<td>0.729</td>
<td>10.178</td>
</tr>
</tbody>
</table>

2.2. Employer's optimisation problem

The employer's problem is to choose the pay structure to maximise the net output index (i.e. the output less remuneration to the employee) subject to the employee's maximisation behaviour outlined above and subject to the constraint that the employee's maximised utility does not fall below a certain value necessary for the employer to remain competitive in attracting employees – i.e. reservation utility. In this section, we first derive the employee's reservation utility function, and then explore the employer's optimisation problem.

One reasonable candidate for the reservation utility could be the utility level associated with the pure piece-rate wage. As in section 2.1, representative employees maximise their utility with respect to $L$, $I$ and $Q$, given the pay structure,

$$\max Y^v(\bar{L} - L)(I - I)^{x\phi}Q^{q\phi}$$

subject to (4) and (3) but with $\alpha = 1$. The first-order conditions require that,

$$\frac{y + x\phi}{X} = \frac{q\phi}{\bar{L}I - X}$$

$$\frac{i}{I - I} = \frac{q\phi L}{\bar{L}I - X}$$

$$\frac{l}{L - L} = \frac{q\phi I}{\bar{L}I - X}$$

solving the nonlinear equation system and substituting the resulted $Y$, $L$, $I$, and $\Phi$ into the utility function, we get the indirect reservation utility function,

$$\bar{U} = w^\nu \left[ \frac{\bar{L}(q\phi + y + x\phi)(y + x\phi)}{(l + q\phi + y + x\phi)(l + q\phi + y + x\phi)} \right]^\nu \left[ \frac{\bar{L}l}{l + q\phi + y + x\phi} \right]^l \left[ \frac{\bar{L}i}{i + q\phi + y + x\phi} \right]^i \times \left[ \frac{\phi q\bar{L}I(q\phi + y + x\phi)}{(l + q\phi + y + x\phi)(l + q\phi + y + x\phi)} \right]^{q\phi} \left[ \frac{\bar{L}(q\phi + y + x\phi)(y + x\phi)}{(l + q\phi + y + x\phi)(l + q\phi + y + x\phi)} \right]^{x\phi}$$

The employer's problem is to choose the pay structure (by choosing $\alpha$, $s$, $w$) to maximise the
net output index subject to the employee’s maximisation behaviour outlined above and subject to the constraint that the employee’s maximised utility does not fall below the reservation utility derived above.

\[
\text{Max: } X^* Q^q - Y \\
\text{s.t. } U^* \geq \bar{U}
\] (14)

By substituting (9) and (13) into (14) and differentiating with respective to \(a\) and \(s\) (taking \(w = 1\), as numeraire), we get the first-order conditions. Solving the equation system, we could get the explicit form of the optimal piece-rate component \(a^*\), and the optimal hourly rate wage \(s^*\) which are dependent on \(y, l, i, \phi, x, \) and \(q\),

\[
\begin{align*}
a^* &= a^*(y, l, i, \phi, x, q) \\
s^* &= s^*(y, l, i, \phi, x, q)
\end{align*}
\] (15)

and \(0 < a^* < 1\). The numerical illustration shows that \(a^* = 0.6371\) when we suppose \(y = l = i = \phi = x = q = 1\), and \(a^* = 0.4823\) if we assume that \(y = 0.4, l = 0.2, i = 0.2, \phi = 0.2, x = 0.5, \) \(q = 0.5\). This result implies that a mixed pay structure instead of either a pure hourly or a pure piece-rate pay maximises the net output index for a given utility level of the employee. Conversely, a mixed pay structure offers the employee a higher utility level for a given output constraint. We thus have,

**Proposition 2:** Ignoring administrative costs, an optimal pay structure is a mixture of piece and hourly rates.

### 3. Concluding remarks

Our results are intuitively agreeable but seem to contradict the empirical observation that a mixed pay structure is not common. It is true that many incentive schemes may be regarded as performing to some extent the piece-rate component. In the long-run, an incentive scheme gives more pay to an employee with higher output. However, why not adopt a formal mixed structure? In his analysis of the status consideration, Frank (1984) provides one factor limiting the use of piece-rate payment. There are probably a number of other factors contributing to the rarity of the mixed structure. Experts in industrial relations and labour economics may be better qualified to provide an explanation.

### Appendix 1

To obtain the comparative-static results, we totally differentiate (3), (5)–(7) and solve the resulting equations to yield the results.

In the system of equations in the text, substitute out \(\Phi\) and \(X\) by using (2) and (4). (We may come back to examine what happens to \(\Phi\) and \(X\) by using (2) and (4) again.) This leaves
us with four equations in four unknowns $Q$, $I$, $L$, $Y$. Totally differentiate these four equations, divide through appropriately to express in proportional form and then rewrite in matrix-vector form to obtain,

$$\begin{bmatrix}
\frac{\alpha w Q}{Y} & A & B & 1 \\
D & E & F & G \\
H & J & K & M \\
N & P & P & R
\end{bmatrix}
\begin{bmatrix}
dQ \\
dI \\
dL \\
dY
\end{bmatrix}
=
\begin{bmatrix}
\frac{w X}{Y} \frac{dW}{w} + C \frac{dA}{\alpha} \\
G \frac{dW}{w} + G \frac{dA}{\alpha} \\
y AwI \frac{dW}{w} + yA \frac{(wI - s)}{Y} \frac{dA}{\alpha} + y(1 - \alpha) \frac{s}{Y} \frac{ds}{s} \\
R \frac{dW}{w} + R \frac{dA}{\alpha}
\end{bmatrix}
$$

where $A = -awIL/Y$, $B = (\alpha wI + (1 - \alpha)sL)/Y$, $C = (\alpha wX - \alpha sL)/Y$, $D = -x\phi LQ/X^2$, $E = IL^2x\phi/X^2$, $F = x\phi LQ/X^2$, $G = yawL/Y$, $H = -x\phi LQ/X^2$, $J = x\phi L^2/X^2 - x\phi LX - yawL/Y$, $K = x\phi L^2/X^2 + IL/(L - L)^2$, $M = [yawI + (1 - \alpha)sL]/Y$, $N = -(q\phi/Q + x\phi Q/X^2)$, $P = x\phi IL/X^2$, $R = yawLY$.

Solve the above system by Cramer’s Rule for $dQ/Q$, $dI/I$, $dL/L$, $dY/Y$ respectively as a function of $dW/w$, $dA/\alpha$, $ds/s$. Rearrange in elasticity form to obtain

$$\frac{(dQ/\alpha)(\alpha/Q)}{Q} = CE(KP - MP) - CJ(FR - PG) + CP(FM - GK) - GA(KR - MP) + GJ(BR - P) - GP(BM - K) + [A(FR - PG) - E(BR - P) + P(BG - F)] + y\alpha(wI - s)/Y - RA(FM - GK) + RE(BM - K) - RJ(BG - F).$$

Appendix 2

To solve the equation system (3), (5)–(7), we first substitute (7) into (5) and (6) and solve for $I$. Substituting the resulted $I$ into (6) and solving for $Q$, we could express the $I$ and $Q$ in terms of $L$, $Y$ and parameters. Inserting the results into (3), we have $Y$. Then, inserting all results back into (7), we have,

$$L = \frac{\phi q + \phi x + y}{\phi q + \phi x + i + y}$$

$$Q = \frac{1}{2} \phi q(\phi q + \phi x + y)\times(2\alpha w + 2\alpha w\phi q + \alpha q - \alpha q + q\psi - q\phi q + \phi q + x\phi + x\phi - x\phi q + \phi q + x\phi + x\phi q + yi + yq\phi)$$

$$Y = -\frac{1}{2} \times(q\phi q - q\phi + x\phi x - x\phi q - x\phi q + \sqrt{f(y, i, i, q, x, q, w, s, \alpha)} - i + 2s\alpha y - yaw - 2y + is\alpha)$$

$$(\phi q - \phi x + y + i)(\phi q + \phi x + 1 + y)$$
\[
I = 1 - \frac{1}{2} i \\
\times \frac{(2aw_i + 2awq\phi + is - is\alpha + q\phi s - q\phi s\alpha + x\phi aw + x\phi s - x\phi s\alpha + yaw + \sqrt{f(y,l,i,\phi,x,q,w,s,\alpha)})}{\alpha wli^2 + 2iq\phi + q^2\phi^2 + x\phi + x\phi^2 q + yi + yq\phi}
\]

where

\[
f(y,l,i,\phi,x,q,w,s,\alpha) = i^2 s^2 + q^2 \phi^2 s^2 - 2i^2 s^2 \alpha + i^2 s^2 \alpha^2 - x^2 \phi^2 s^2 + y^2 \alpha^2 w^2 + 2\phi awixs \\
- 2\phi s^2 wixs + 2q\phi^2 \alpha wxs - 2q\phi^2 \alpha^2 wxs + 2iq\phi s^2 - 4iq\phi s^2 \alpha + 2iqs^2 x \\
- 4iqs^2 x\alpha - 2isyaw + 2iqs^2 \alpha^2 + 2iqs^2 x + 2is\alpha^2 yw - 2q^2 \phi^2 s^2 \alpha \\
+ 2q\phi s^2 x - 4q\phi^2 s^2 x\alpha - 2q\phi syaw + q^2 \phi^2 s^2 \alpha^2 + 2q^2 \phi^2 s^2 \alpha^2 x \\
+ 2q\phi s^2 yw + x^2 \phi^2 s^2 w^2 + 2x^2 \phi^2 \alpha w^2 - 2x^2 \phi^2 \alpha^2 ws \\
+ 2x\phi^2 w^2 y - 2x^2 \phi^2 s^2 \alpha + 2x\phi syw + x^2 \phi^2 s^2 \alpha^2 - 2x\phi s\alpha^2 yw .
\]

References


