The Economic Theory of Clubs: Optimal Tax/Subsidy

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In a previous paper (1973) I derive the Pareto optimality conditions for clubs (i.e. public goods in which membership affects preferences). In Section I of this paper I compare these optimality conditions with the equilibrium conditions, assuming club members maximize net benefit per person. The equilibrium size of a club is smaller than the optimal size: the optimal tax/subsidy to achieve Pareto optimality is derived accordingly in Section II. The rate of subsidy increases at a decreasing rate over the relevant range. In Section III a complication due to the difficulty of attracting sufficient members is discussed. This difficulty arises because, given equal cost-sharing by all members, the benefit derived by the marginal member cannot fall short of the average cost (per person). The optimal tax/subsidy derived in Section II proves to be still applicable; the equilibrium club size is usually but no longer necessarily smaller than the optimal size.

I. COMPARISON OF EQUILIBRIUM AND OPTIMAL CLUB SIZES

Buchanan (1965) makes an important contribution to the literature on public goods by recognizing the fact that the number of individuals consuming a public good may affect the satisfaction of the individuals. Hence, the utility of each individual is not only a function of the amounts of the various goods (private and public) consumed but also a function of the number of individuals \( N_j \) in each “club” (composed by those individuals consuming each public good). This is a more realistic analysis since for most pure and semi-public goods such as swimming pools, museums, associations, journals, etc., the number of consumers does affect one’s satisfaction one way or another due to congestion, prestige, desire to mix with crowds, etc. However, in deriving his “statements for the necessary marginal conditions for Pareto optimality”, Buchanan maximizes the utility function of an individual subject to his own feasibility constraint. As a result, Buchanan obtains the following condition,

\[
U_{x_j} / U_{x_i} = f'_{x_j} / f'_{x_i}
\]

which states “that the marginal rate of substitution ‘in consumption’ between the size of the group sharing in the use of good \( X_j \), and the

1 I am grateful to the referees of this and the previous papers for inspiration and comments. This paper was finalized during my visit to the London School of Economics under a Nuffield Foundation Fellowship.
numeraire good $X_r$, must be equal to the marginal rate of substitution "in production". With the further assumptions of identical individual taste and symmetrical cost sharing, this condition results in the maximization of average net benefit (net benefit per person) as Buchanan shows in his geometry. In my previous paper I show that, by using a proper procedure of Pareto maximization, the following optimality condition can be derived,

\[ \int_0^{X_j} \left( \frac{U_{X_j}}{U_{X_n}} \right) dX_j + \sum_{k \neq j} \frac{U_{X_j}}{U_{X_n}} \geq 0 \text{ if } i \in S_j; \quad \leq 0 \text{ if } i \notin S_j \]

where $X_j$ is the quantity of the public good $j$ and $S_j$ is the set of indices corresponding to individuals in club $j$. This condition says that, for each public good, any individual in the club must derive a total benefit from the consumption of that good in excess of (or at least equal to) the aggregate marginal disutility imposed on other consumers in the club, and that the reverse must hold for any individual not in the club. This condition leads to the maximization of aggregate net benefit as well as the attainment of Pareto optimality. To simplify discussion below, I shall speak mainly in terms of average and aggregate benefit instead of the marginal rate of substitution (MRS). (The existence of a pure private good consumed by all individuals allows us to express the marginal utilities and disutilities in terms of the MRS. If we use this private good as our monetary unit, these MRS are transformed into marginal benefits measured in monetary terms. We cannot express our optimality conditions in these simple and convenient terms of MRS or benefit if we do not have this pure private good; see Ng, 1972.)

Buchanan's condition, rather than being the Pareto condition, is more appropriate as the market equilibrium condition. Assuming that club members maximize average benefit, this condition is the first-order condition for the point from which there is no incentive for the club to expand or contract its membership. With appropriate second-order requirements, this equilibrium point is also stable. In practice, what a club attempts to maximize of course varies from case to case. If the public good is owned by a private individual or firm, it is most likely that profit will be maximized. (With profit maximization, optimality will be attained if some classical conditions such as many competitors, non-externality, etc., are satisfied. In the case of clubs, however, it is more likely that some monopolistic element will be present.) If the public good is provided on a cooperative basis, then average benefit will more probably be maximized. But the operation of a club is usually controlled by the executive committee which consists mainly of some old members. It is possible that they may put more weight on their own interest. But the assumption of average benefit maximization seems to be a good approximation. That the self-interest of club members will lead to the maximization of average (net) benefit is based on the assumption of equal cost-sharing by all members. This assumption also seems to be quite acceptable since discrimination in membership fees is usually imperfect and mainly designed for purposes other than the full
exploitation of consumers' surpluses. In the following I shall assume equal cost-sharing. Divisibility, continuity, and the fulfillment of second-order conditions are also assumed.

Notation: 
- \( A \) = aggregate benefit
- \( T \) = total cost
- \( N \) = number of members (assumed large and hence taken as roughly continuous)
- \( X \) = quantity (and/or quality) index of the public good
- \( B = A/N \) = average gross benefit (per person)
- \( C = T/N \) = cost per person
- \( E = A - T \) = aggregate net benefit
- \( M \) = gross benefit experienced by the marginal member.

We shall deal with the more general case in which both \( A \) and \( T \) are functions of both \( X \) and \( N \); a subscript denotes partial differentiation. The case of a Samuelsonian public good can be obtained by putting \( T_N = 0 \) for all values of \( N \) (more accurately, no inter-membership externalities are also required).

Pareto optimality requires the maximization of aggregate net benefit \( E \), giving the following first-order conditions,

\[
\begin{align*}
(3) & \quad A_X - T_X = 0 \\
(4) & \quad A_N - T_N = 0
\end{align*}
\]

or the equality of marginal benefit and marginal cost with respect to both variables \( X \) and \( N \). On the other hand, the maximization of average net benefit \( (A - T)/N \) gives

\[
\begin{align*}
(5) & \quad A_X - T_X = 0 \\
(6) & \quad A_N - T_N = (A - T)/N.
\end{align*}
\]

Since (5) is identical with (3), it means that, given the number of club members \( (N) \), the maximization of average net benefit leads to exactly the same quantity of the good as the maximization of aggregate net benefit. Hence the deviation from Pareto optimality arises from the "inappropriate" choice of \( N \), not from the choice of \( X \) as such. This "inappropriate" choice of \( N \) is indicated in (6) where the marginal net benefit is equated with the average net benefit, a condition for the latter to be at its maximum point. But optimality requires the equating of marginal net benefit to zero, as indicated in (4). It can be shown that this deviation is a downward one (involving a smaller \( N \) than optimal).

Examine Figure 1, where the various curves have been clearly labelled and need no explanation. Due to the identity of equations (3) and (5), we can disregard the variable \( X \) and operate with the two-dimensional Figure 1. For each level of \( X \), we have typically a situation depicted therein.\(^1\) The Pareto optimal level of \( N \), \( N_p \), is determined by the intersection of the curve \( E_N \) with the horizontal axis, and the level of \( N \) chosen by a club maximizing average net benefit, \( N_\alpha \), is determined by

\(^1\) If \( T_N = 0 \), the curve \( C \) becomes a rectangular hyperbola, and \( E_N \) cuts \( B \) at the latter's maximum point. This particular case is not assumed in the text.
the intersection of $E_N$ and $B-C$ at $G$. $G$ is, of course, the maximum point for $B-C$, where the slope of $B$ equals that of $C$. To see that $N_c < N_p$, note that $B-C$ must be positive at $N_c$ for the club to be viable. Moreover, for this to be the maximum instead of the minimum point, $E_N$ must cut $B-C$ from above and be falling. Hence $N_p > N_c$ for the case of well-behaved curves depicted in the figure. It is not difficult to realize that the reverse cannot hold even if the curves are not well-behaved or if we have multiple maxima. For the global maxima, $N_p$ must still be larger than $N_c$. The reverse cannot be true because if $N$ is smaller than $N_c$, we can always increase the aggregate net benefit ($E$) by increasing $N$ right to $N_c$, where the average net benefit is highest. Usually we can increase $E$ further by increasing $N$ beyond $N_c$ unless there is a serious kink or discontinuity in the curve, in which case $N_p$ can possibly be equal to $N_c$. Hence, $N_c \leq N_p$ is a general result. To avoid using the clumsy expression "smaller than or equal to", I shall disregard below the (unlikely) case of equality.

II. OPTIMAL TAX/SUBSIDY

Since the equilibrium size of a club is smaller than the optimal size, there arises the welfare question of increasing the size of clubs. The potential gain of increasing club size from $N_c$ to $N_p$ is measured by the area $GN_cN_p$ in Figure 1. But how do we get to $N_p$? Mutual agreement or bargaining between existing and prospective club members is ruled out by the assumption of equal cost-sharing. Governmental decrees with respect to the sizes of clubs do not square well with the liberal tradition. This leaves us with the tax/subsidy as an obvious instrument.
to achieve optimal club sizes. I turn now to derive the optimal tax/subsidy, $S^*$. ($S$ is a subsidy if positive and a tax if negative.)

With payment of $S$ from the government, the aggregate net benefit of a club is now $E+S$. Maximization of $(E+S)/N$ gives

(7) $E_X + S_X = 0$

(8) $(E_N + S_N)/N - (E+S)/N^2 = 0$.

Since Pareto optimality requires that $E_X = 0$ and $E_N = 0$, $S^*$ must be so selected that when (7) and (8) are satisfied, both $E_X = 0$ and $E_N = 0$. Hence, the optimal tax/subsidy can be derived by equating the left-hand side of (7) to $E_X$ and that of (8) to $E_N$, leading to

(7') $S^*_X = 0$

(8') $(E_N + S_N^*)/N - (E+S^*)/N^2 = E_N$.

Equation (7') means that the optimal tax/subsidy is independent of $X$. Hence, $S^*$ is a function of $N$ alone. This means that (8') is a first-order differential equation, the solution of which yields

(9) $S^* = (E+K)N - E$

where $K$ is the constant of integration and can be of any positive or negative value, provided it remains a constant, without affecting the first-order condition. However, we need only to have some common-sense to realize that, if $K$ is negative, it must not be so large in absolute value as to make the average benefit negative even if the club has attained its optimal membership, and hence drive the club out of existence. To avoid this latter outcome, we have to impose the constraint that $E^* + S^* > 0$, where $E^*$ is the maximal value of $E$, which, of course, occurs at $N_p$. Substituting this into (9), we have $K > -E^*$. Hence, we may state the requirement for $S^*$ more completely as

(9') $S^* = (E+K)N - E$ subject to $E^* + K > 0$.

This $S^*$ can be regarded as a positive subsidy $(E+K)N$ plus a tax on economic surplus of the club, $E$. The economic surplus is to be taxed away in toto so as to kill the incentive for restrictive behaviour and the positive subsidy provides incentive to follow the optimal policy.

Let us examine now the change in the subsidy $S^*$ and the rate of subsidy $S^*/N$ with respect to $N$.

(10) $dS^*/dN = E + K + (N-1)E_N$

(11) $d(S^*_N)/dN = (1 - 1/N)E_N + E/N^2$.

The sign of (10) cannot be established due to the constant term $K$ which may be negative. But the right-hand side of (11) is positive over the relevant range where both $E$ and $E_N$ are positive. Since

$$\frac{dS^*}{dN} = N d\left(\frac{S^*}{N}\right)/dN + \frac{S^*}{N},$$

we can infer from the positivity of $d(S^*/N)/dN$ that $dS^*/dN$ must also
be positive unless $S^*$ itself is negative. Hence, if we have a real (positive) subsidy, both the subsidy and the rate of subsidy increase with $N$ before the optimal point $N_p$ is reached. If the subsidy is negative, i.e. really a tax, then the rate of tax decreases with $N$, but the tax itself may increase or decrease with $N$.

At first sight it may seem intuitive that, since the objective is to encourage the club members to stick to the optimal point $N_p$, the rate of subsidy should decrease after $N_p$, so that the curve $S^*/N$ has a peak at $N_p$. This, however, is not true, as can be seen from (11). Immediately to the right of $N_p$, $E_N$, though negative, is small in absolute value. Hence the positive term $E/N^2$ in the right-hand side of (11) dominates, indicating that $S^*/N$ is still an increasing function of $N$. This can be seen more vividly as shown below.

The curve $S^*/N$ is depicted in Figure 2, which is similar to Figure 1 but the curves $B$ and $C$ have been omitted. The club attempts to maximize $(E + S)/N$. Hence, $S$ must be such as to make $(E + S)/N$ attain its peak at $N_p$. But since $E/N$ is declining at this point, $(E + S)/N$ will also be declining if $S/N$ attains its peak at $N_p$. This will cause the club to aim at a smaller membership than $N_p$. To make $(E + S)/N$ attain its maximum point at $N_p$, the slope of $S/N$ has to offset that of $E/N$. Hence, at the point $N_p$, $S^*/N$ is increasing and has an absolute slope equal to that of $E/N$.

When does $S^*/N$ itself attain its peak? Put it differently, after which point should the rate of subsidy be reduced? To discover this point, put the right-hand side of (11) equal to zero. This gives us the condition $E_N = d(E/N)/dN$; $S^*/N$ is maximized at the point $N_s$ where the slope of $E/N$ is equal to the value of $E_N$. The intuitive explanation for this is that, when the slope of $E/N$ is equal to the value of $E_N$, we do not need
any additional incentive or disincentive measure. The club, in trying to 
maximize $E/N$, equalizes its slope to zero. And the optimal point $N_p$ 
ocite{314}occurs when $E_N$ equals zero. Hence, if the slope of $E/N$ happens to 
equal the value of $E_N$, we make the rate of subsidy constant.

The slope of $S^*/N$, as drawn in Figure 2, is decreasing. To show that 
this is the case, differentiate (11) to get

\[ (11') \quad d^2\left(\frac{S^*}{N}\right) = \left(1 - \frac{1}{N}\right)E_{NN} + \frac{2}{N^2}\left(E_N - \frac{E}{N}\right). \]

Thus, over the relevant range where $E/N$ is larger than $E_N$ and where $E_N$ is declining, the right-hand side of (11') is negative, implying that the slope of $S^*/N$ is decreasing.

We have thus shown that the optimal rate of subsidy is an increasing 
function (one that increases at a decreasing rate) over the relevant range 
between $N_C$ and $N_P$. The total subsidy can easily be calculated from $S^*/N$.

It may be unnecessary but prudent to remember that the $S^*/N$ curve 
drawn in Figure 2 is only one out of many possible curves that satisfy 
the optimality requirement. Due to the constant term $K$, $S^*/N$ can be 
moved vertically upwards or downwards, provided that $(E+S*)/N$ 
is not made wholly below the horizontal axis.

A qualification to the above analysis must be mentioned. If we allow 
discontinuity in the curve $S^*/N$ (continuity has been assumed in this 
paper), then $S^*/N$ need not be increasing after the point $N_P$. It could 
follow the path as depicted in Figure 2 up to $N_P$ and then suddenly kink 
downwards or drop to a low value. $(E+S*)/N$ would still have a peak 
at $N_P$. This solution, however, has a drawback. When the estimated 
optimal membership point diverges from the true optimal point, e.g. 
due to an overestimation of the slope of $E/N$, our smooth solution 
results in a point in between. But the discontinuous solution usually 
results in the estimated optimal point, and hence is inferior.

III. A COMPLICATION—AVERAGE COST CANNOT 
EXCEED THE BENEFIT OF THE MARGINAL MEMBER

Our analysis so far is based on the implicit assumption that the club 
has no difficulty in attracting the desired number of members. However, 
with the assumption of equal cost-sharing, each member, in maximizing 
his own utility, must benefit from joining the club by an amount no less 
than the average cost $C$. Those who benefit less will not join. Hence there 
is the very likely possibility that a club may not be able to attract the 
number of membership desired, given equal cost-sharing. We have, in 
effect, the constraint $M \geq C$, where $M$ is the benefit of the marginal club 
member. It must be noted that $M$, the benefit of the marginal club 
member as an individual, is different from $A_N$ or $A_N - T_N$. Now

\[ A = \sum_{i=1}^{N} M^i. \]
Hence

\[ A_N = M + \sum M_N. \]

So

\[ M = A_N - \sum M_N. \]

Thus \( M = A_N \) only if the addition of this marginal member does not affect the benefit of existing members. Put it differently, the existence of inter-membership externalities (usually positive at first, then negative) implies that the marginal benefit to the club with respect to membership differs from the benefit experienced by the marginal member.

(i) *Equilibrium Conditions*

If \( M > C \) at the position of equilibrium, then the solution is the same as (5) and (6). To examine the case where the constraint is effective, assume that equality holds at the position of (constrained) equilibrium.

\[ (12) \quad M - C = 0. \]

Maximizing \( (A - T)/N \) subject to (12) we have the following condition in addition to (12) itself:

\[ \frac{(A_N - T_N)/N - (A - T)/N^2}{M_N - C_N} = \frac{(A - T)/N}{M - C}. \]

On the left-hand side the numerator is the difference between marginal and average net benefit, divided by \( N \). The denominator is the change in average net benefit by changing \( X \). Hence the left-hand side as a whole is the effect on average net benefit of a change in \( N \) relative to that of a change in \( X \). This ratio can be taken as the MRS between \( N \) and \( X \) as average net benefit is the objective to be maximized. On the other hand the right-hand side of (13) is the effect on the constraint of a change in \( N \) relative to that of a change in \( X \). Since \( N \) and \( X \) are constrained by (12), this relative effect on the constraint is their MRT (marginal rate of transformation). This transformation is not in the physical sense of production but in the sense that, if one is increased, the other has to be reduced by such a ratio as to satisfy the constraint. Thus equality (13), being the traditional equality of MRS and MRT, can be seen to be intuitively reasonable.

Now both \( M_X \) and \( C_X \) are usually positive and the latter is likely to be larger than the former in the relevant range. This is so because \( B_X = C_X \) at the unconstrained equilibrium position, and \( M_X \) is likely to be smaller than \( B_X \). If this (i.e. \( M_X - C_X < 0 \)) is so, then \( X \) has to be reduced from its unconstrained equilibrium position to meet the constraint, and \( A_X > T_X \). If the reverse (i.e. \( M_X - C_X > 0 \)) is true, then \( X \) has to be increased, and \( A_X < T_X \). In either case the denominator of the left-hand side and that of the right-hand side are of opposite signs. The numerators must therefore also be of opposite signs. This can be seen more vividly in Figure 3.
In the present case of constrained maximization we really have to operate with a three-dimensional figure. But since this is very complicated I shall assume that $X$ has been fixed at the desired level and concentrate on the variable $N$. In Figure 3 the unconstrained equilibrium point of the club is at $N_c$ where $(A-T)/N$ is maximized. If the $M$ curve is the first one, then $M > C$ at this equilibrium point. The constraint is ineffective and no adjustment is necessary. At this juncture it must be pointed out that we can operate with different $M$ curves with the same set of $(A-T)/N$ and $C$ curves because of the effect of $\sum M_i$ which may have different values at different levels of $N$. Of course, these different $M$ curves do not exist simultaneously but represent alternative possibilities. Now, if the curve is $M_2$, $M < C$ at $N_c$, and the curve $M_2$ is steeper than the curve $C$, implying $|M_N| > |C_N|$, and $M_N - C_N < 0$, $N$ has therefore to be reduced to $N_2$ to make $M = C$. The reduction of $N$ from $N_c$ means, however, that $A_N - T_N$ will be made larger than $(A-T)/N$. This corresponds with the requirement that the numerators of (13) be of different signs. On the other hand if $M_3$ holds, $M_N - C_N > 0$ as $C$ is steeper than $M_3$. So $N$ has to be increased to $N_3$ and $A_N - T_N < (A-T)/N$. This still satisfies the requirement of opposite signs for the numerators.

From the figure it can also be seen that the constrained equilibrium point $N_2$ ($N_3$) is further away from (closer to) the optimal point $N_p$ than the unconstrained equilibrium point. But we have not taken the variable $X$ into account. The constrained equilibrium may involve a
different level of $X$ and hence make the point $N_2$ (or $N_3$) not directly comparable with $N_p$ on the two-dimensional Figure 3. However, as previously pointed out, the constrained equilibrium point is likely to involve a lower level of $X$ than the unconstrained equilibrium. And a smaller $X$ is usually associated with a lower equilibrium level of $N$. Hence $N_2$ is likely to be further away from $N_p$ if we take the effect of different levels of $X$ into account. Even if the constrained equilibrium point involves a higher level of $X$ and hence a corresponding $N$ closer to or even coincident with $N_p$, it does not represent an optimal solution. With a higher given value of $X$, the corresponding optimal $N$ is then of higher value. The overall optimal solution is of course $N_p$ and its corresponding level of $X$.

Nevertheless $N_3$ is to the right of $N_p$ and it might even lie to the right of $N_p$. This is shown in Figure 4 as $N_4$. This will be the case if $M$ is very high initially, drops rather rapidly to flatten off and intersects $C$ from below at a point right of $N_p$. This means that there are some inframarginal members who derive very high benefits from the club, not many members with intermediate benefits and a lot more with moderate benefits. The negative effect of number is rather high at a relatively low level of $N$ so that $(A - T)/N$ is maximized at relatively low $N$. But at this point the per person cost is too high for the marginal member. Membership has therefore to be increased to lower $C$. Since $M$ does not fall rapidly at this range the requirement $M \geq C$ may then be achieved at a higher $N$ ($N_4$ in Figure 4).
A sufficient condition to ensure that, for each level of \( X \), \( N_p \) is necessarily no smaller than the (constrained) equilibrium club size is: the marginal cost \( T_N \) is not larger than the average cost \( C \) and the aggregate inter-membership externality \( \sum M_k \) is non-negative over the relevant range. Since \( M = A_N - \sum M_k \), the second part of the condition ensures that \( A_N \geq M \). But \( M \geq C \) by the constraint. Together with the first part of the condition, we have \( A_N - T_N \geq 0 \). But if this is so, total net benefit can be increased by increasing \( N \) up to if not beyond the equilibrium value. Hence \( N_p \) cannot be smaller than the latter. The first part of the above sufficient condition is usually satisfied for most public goods; if the public good is pure in the Samuelsonian sense, \( T_N \) is in fact zero. Hence an excessive equilibrium club size is usually associated with a strong and negative inter-membership externality. Being based on subjective preference, this external effect \( \sum M_k \) is difficult to evaluate. For the club “Royal Economic Society”, or “American Economic Association”, for example, an increase in membership may raise the satisfaction of some members eager to belong to a large and authoritative association, but the reverse may be true for those preferring an exclusive club. But if we have reason to believe that the positive effect predominates, then we can be pretty sure that the unsubsidized equilibrium size is smaller than the optimal.

A question arises: if the curve is \( M_3 \) in Figure 3, will the club be able to expand to \( N_3 \) ?\(^1\) After reaching the point \( N_0 \), where \( M_3 \) intersects \( C \) from above, the club is viable if the net benefit is positive. The point \( N_3 \) is also viable and superior if average net benefit, \( (A - T)/N \), is higher. In our static analysis the club will of course expand at one stroke to \( N_3 \) by reducing per member cost to the appropriate level and attracting sufficient members instantaneously. However, in reality, the expansion process is usually a slow one. But each intermediate step between \( N_0 \) and \( N_3 \) involves a higher cost per member than the benefit to the marginal member. The club, if it wishes to expand towards \( N_3 \), may therefore have to sustain initial losses financed from reserves or borrowing. This borrowing can be repaid by charging a slightly higher fee than that indicated by \( C \) after expansion to \( N_3 \) (actually a little to the right of \( N_3 \) due to the extra fee). Alternatively the club may abandon, at least temporarily, the principle of equal cost-sharing and offer its new members a lower fee. This may in fact be the reason why so many journals, magazines and associations occasionally offer concession subscription rates to new members for certain limited periods.

It is also possible that the club will not be viable with equal cost-sharing without government subsidy even though \( (A - T)/N \) is positive over a large range. This will be the case if the cost curve (such as \( C' \) in Figure 4) lies entirely above the \( M \) curve. Though this failure of the market may seem similar to the case of decreasing costs, it may happen even if there are no decreasing costs in the production of \( X \).

\(^1\) I owe this question to E. J. Mishan.
(ii) **Optimal Tax/Subsidy**

Now consider the tax/subsidy necessary to achieve optimality for the case where the constraint $M \geq C$ is effective. A tax/subsidy $S$ causes the maximand and the constraint to be revised to

$$\text{(14)} \quad \frac{(E+S)}{N},$$

$$\text{(15)} \quad M + \frac{S}{N} - C \geq 0.$$

Maximizing (14) subject to (15) requires, disregarding the case where either $X$ or $N$ (or both) equals zero, which implies the non-existence of the club, that

$$\text{(16)} \quad (E_N + S_N)/N = -\theta \left( M_N + \frac{S_N}{N} - C_N \right)$$

$$\text{(17)} \quad \frac{E_N + S_N}{N^2} = -\theta \left( M_N + \frac{S_N}{N^2} - C_N \right)$$

$$\text{(18)} \quad \theta \geq 0, \quad M + \frac{S}{N} - C \geq 0, \text{ and } M + \frac{S}{N} - C = 0 \text{ or } \theta = 0,$$

where $\theta$ is the multiplier associated with (15). If $\theta$ equals zero, (16) and (17) are exactly the same as (7) and (8). In that case our solution of $S^*$ contains an arbitrary term $K$. Hence we can always select a value of $K$ to make $M + (S/N) - C > 0$ and hence $\theta = 0$. An obvious solution for optimal $S$ is therefore equation (9') with the additional constraint $M + (S/N) - C > 0$, or $K > (E/N) - E + C - M$. Since the imposition of this $S^*$ will enable the optimality conditions (3) and (4) to be satisfied, the values of $E$, $C$, etc. in the preceding constraint refer to the values at the optimal position. Moreover, it does not matter if we make this inequality constraint an equality as it applies to the optimal position and hence does not reduce the maximand. $\theta$ will become non-zero only if the constraint effectively reduces the maximand. Hence our solution of $S^*$ may be stated fully as

$$\text{(9')} \quad S^* = (E + K)N - E$$

subject to $E^* + K > 0$ and $K \geq (E^*/N) - E^* + C^* - M^*$.  

This last constraint on the constant $K$ implies that $S^*$ is more likely to be positive. As long as $C > M$ at the optimal position, $S^*$ must be positive; if $M > C$, then $S^*$ could be made negative, but it is always feasible to make $S^*$ positive irrespective of the sign of $M - C$. 

It must be admitted that (9') is only one of the possible solutions of $S^*$. In other words there may be other forms for $S$ that satisfy the optimality requirement. This is so because the solution of partial differential equations involves arbitrary functions rather than arbitrary constants in the case of total differential equations. In solving for $S^*$, I use the trick of borrowing the solution from (9') instead of going through (16) and (17) because the latter approach involves several integrals which prove difficult to evaluate.
Another two qualifications to the analysis above may also be mentioned. First, the analysis assumes that second-order conditions are satisfied. Apart from the requirement of "well-behaved" benefit and cost curves, this means that individuals with larger benefits are admitted to the club before individuals with smaller benefits. In practice, this may be difficult to enforce, and something like the "first come first served" principle usually applies. If this consideration is taken into account, efficiency requires that the subsidy should not be so large as to unduly encourage individuals with negative social net benefit to join the club. Secondly, the use of a fixed $M$ curve involves a simplification as each $M^i$ is not, in general, invariant with respect to $N$. This simplification does not affect the essence of the issue. Alternatively, the $M$ curve may be interpreted as the benefit to the marginal consumers at the respective levels of $N$. Thus interpreted, $A$ is not necessarily equal to $\Sigma M^i$ and $M$ is not necessarily monotonically decreasing. But again, nothing of substance is affected.\footnote{I have benefited from a seminar discussion at the University of York in which Mr M. W. Jones-Lee pointed out to me that, in my previous paper, instead of putting $D_{ij}$, the variable indicating membership, independently into the utility function, it could be more appropriately introduced jointly with $X_i$ and $N_i$ so that instead of $X_i, N_i$ and $D_{ij}$, we have $X_iD_{ij}$ and $N_iD_{ij}$ as arguments. This revision does not affect the result, namely equation (2) above, which could be generalized by replacing the zeros with $F_{N_iF_{X_i}}$ to account for the case of non-zero costs of increasing membership.}

IV. CONCLUDING REMARKS

Our analysis indicates that, to introduce an optimal tax-subsidy for a club, we must know at least the net benefit function $E$. It is true that the exact knowledge of this function is very difficult to obtain. But the same thing applies to the benefit or damage function of an external effect, the knowledge of which is required for the corresponding corrective tax/subsidy. In the real world, where perfect knowledge is not available at negligible costs, we usually have to base our decision on a rough estimate which is, however, better than no estimate or random decisions. With the same degree of accuracy in our estimates, we will do better with a good theory than with no theory or bad theories. Moreover, the theory of clubs need not be used only for prescriptive purposes but may also be used for descriptive purposes. For example, it may explain certain immigration policies followed by many countries in the world. And, as pointed out above, it may explain the concession rates offered by periodicals and associations. Furthermore, the economic theory of clubs is quite general and applies in various degrees to most institutions where membership is an important variable. This ranges from families to swimming pools, from a club of a few persons to the United Nations. Hence the theory of clubs seems worthy of development. The theory I have been developing so far is based on some simplifying assumptions; thus one possible line of development lies in the relaxation of these
assumptions. For example, one of the assumptions used so far is that the utility of each individual is only affected by the number of individuals in the club but not by the composition of the membership; each individual does not discriminate in favour of or against any other individual. The relaxation of this assumption leads to an optimality condition slightly different from (2) but analysis similar to that developed in this paper can be applied to this new condition. As another example, we have been assuming that an individual is either admitted fully into the club or is excluded. A more general model can be constructed in which there may be full or partial consumption of the public good. This generalization results in an optimality condition which says that, if an individual consumes the full amount of (part of; no amount of) the public good, the marginal benefit he himself derives from the good must exceed or equal (equal; equal or fall short of) the aggregate net costs (i.e. disutilities measured in terms of the numeraire) suffered by others caused by his marginal consumption. With this generalization, however, it becomes difficult to apply the analysis of optimal tax/subsidy. With the possibility of different degrees of consumption, the concept of membership is made ambiguous and it is even less certain what a club attempts to maximize.

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