Geometric nonlinear equivalence of frictional systems for compensation

T. Tjahjowidodo
Nanyang Technological University, School of Mechanical and Aerospace Engineering
50 Nanyang Avenue, Singapore 639798
e-mail: ttegoeh@ntu.edu.sg

Abstract
Being a complex nonlinear phenomenon, friction is a difficult part to characterize and identify in a mechanical system. Friction is the result of interaction between one body over or along another and is dependent on many parameters, such as contact geometry, topography, surface materials, presence and type of lubrication and relative motion. The classical Coulomb friction has been widely used especially in the field of control engineering to compensate for the static force in a system. However, despite its simplicity, the effectiveness of the Coulomb model in capturing the frictional behavior in the presliding regime is relatively low. Further study shows that friction force exhibits strong (hysteretic) nonlinear relationship with sliding velocity, displacement and time, which makes the characterization become a difficult task to fulfill. As a consequence, an advanced control strategy, if necessary from the nonlinear class, is necessary to compensate for the effect of friction in high-precision positioning devices.

This paper deals with the development of efficient (nonlinear) control structures, which are optimized based on the geometric nonlinear equivalent system that dynamically represents the nonlocal memory hysteresis friction in geometric form, i.e. an equivalent system with geometric damping and spring. The equivalent system is analyzed using the skeleton technique, which employs the instantaneous amplitude and frequency of the response output of the system under free vibration condition. The results show that the controllers are able to compensate for friction in the system, which also confirms that the equivalent system is efficient to mimic the dynamic behavior of the frictional system.

1 Introduction

Various control structures from linear and nonlinear classes have been tried to deal with mechanical systems with friction element. In a linear control class, proportional-derivative (PD) and proportional-integrative-derivative (PID) actions, which are the most common in industrial applications, are suitable to compensate for friction in macro-level displacement. Introducing an integrative action in the controller may minimize the steady state error of the system, however, this action results in a slower response of the system and might reveal limit cycle problem [1]. On other hand, without integrative action, a PD controller may exhibit a higher steady state error, even though, increasing the derivative (D) gain can eliminate the stick-slip problem due to increased damping in the system.

Another approach in the class of linear controllers is a combined speed and position loop, which usually referred to as a cascade controller. It is among simple controllers that popularly deal with mechanical systems with friction. In this controller, the innermost force loop is enclosed by a speed control loop, and further enclosed by the position loop. Commonly, the controller utilizes a PI controller in the speed loop and P control action in the position loop. The integrative (I) action will eliminate the steady state speed error and the P gain in the speed loop will behave similarly to the derivative action in the position loop. Therefore, the tuning parameters in this controller may be more elaborate than that with a single stage control structure (using a position loop only). In order to improve the tracking performance of the system
with cascade controller, the desired velocity (and/or acceleration) is commonly fedforward to the plant. Cascade and feedforward strategies are commonly applied to systems, whose major objective is to improve rejection of disturbances, which in this case are friction forces (see [2]). From nonlinear control class, the backstepping control strategy has been widely explored to compensate for frictional effect in mechanical systems. Some researchers Friedland studied and developed an adaptive backstepping controller for this purpose (e.g. [3],[4],[5],[6]). However, the objective of those studies is mainly to deal with macro-level displacement system, in which the friction force is modeled using a conventional Coulomb friction.

Moving toward a micro-positioning system, compensation for frictional effect is not straightforward. When a motion of a contacting body is reversed until certain distance from the reversal point, the frictional effects of the mechanism become highly nonlinear. In this stage, which is commonly called as a presliding regime, friction appears predominantly as a function of position. Furthermore, the function does not only depend on the input and output at some time instant in the past, but also on some past extremum values of the input and output, where particularly involves a so-called non-local memory hysteresis function (see [7],[8],[9],[10]).

However, when the system exceeds the stiction (pre-sliding) distance, it will enter the sliding regime. At this stage the friction will predominantly behave as a function of velocity. Generally, the friction force at sliding regime is determined by the steady state function of friction in velocity, namely the Stribeck function. Commonly, a Stribeck function is a decreasing function bounded by a static friction at zero velocity and Coulomb friction at high velocity.

If an accurate model of the system is available, a compensation of the error in the system can be easily made by applying a force command that is equal and opposite of the instantaneous actual force. The strategy of this model based compensation relies on using a measured state as input to the model of the system, which makes the system compensated in feedback loop. The use of desired reference input to be fed onto the model is also common in industrial applications, [11]. However, this control strategy is very sensitive to the model of the system.

The Discontinuous Nonlinear Proportional Feedback (DNPF) controller is proposed by Southward et al., [12]. The controller adds an extra compensating torque, which is slightly higher than the Coulomb force when the position error is small enough to lie within the presliding regime. The stability of this control strategy can be ensured by taking the Lyapunov function candidate ([1]). However, the output of a system with the DNPF controller might reveal a chattering problem due to the discontinuous control input around zero position error. In order to avoid the chattering problem in the result, a smoothing technique can be introduced in this compensation scheme by performing a sigmoid-like function altering the discontinuous control around zero position error. A finite slope in the sigmoid-like function around zero position error is expected to reduce the ‘nervous’ chattering of the response. Similar to the DNPF, Cai and Song, [13], proposed PD controllers with a Smooth Robust Nonlinear Feedback (SRNF) compensator that adds a nonlinear sigmoid-like function to provide an additional stiction force to the control input.

Another potential control structure to deal with a mechanical system with friction is the sliding mode technique ([14]). Modeling inaccuracies can be classified into two major kinds: structured (or parametric) uncertainties and unstructured uncertainties (or unmodeled dynamics) and these can have strong adverse effects on nonlinear control systems. One of the most important approaches to deal with model uncertainty is a robust control. The typical structure of a robust controller is composed of a nominal part, similar to a feedback control law, and additional terms aimed at dealing with model uncertainty. Sliding mode control is one of the important robust control approaches (see e.g. [15],[16]). Tjahjowidodo ([17]) presented the implementation of the sliding mode controller to compensate for friction by treating two different regimes of friction with two separate sliding planes.

Despite many attempts in the development of controllers to compensate for friction in micro-level displacement, it is clearly seen that the difficulty lies in the complex behavior of the friction itself, which comprises the nonlocal memory hysteresis function of displacement in the presliding regime. In order to investigate the complex dynamic behavior of friction, Tjahjowidodo ([18]) proposed a geometric nonlinear equivalence of a mechanical system subjected to frictional surface that is developed via skeleton technique. When a geometric nonlinear system is freely vibrating, the vibration response of the system...
contains the information of dynamic parameters of the system. Extracting the instantaneous amplitude and frequency of the response signal, the modal parameters (e.g. the stiffness and damping function) can be constructed (see [19]). Treating a mechanical system with frictional element, - which basically manifests itself as a material nonlinear system -, as a geometric nonlinear system, the geometric nonlinear equivalence can be constructed.

This paper deals with the evaluation of (nonlinear) control structures that are developed based on the geometric equivalent dynamic parameters, namely the stiffness and the damping, of mechanical systems with material nonlinearity. In particular, this paper discusses different control structures, i.e. feedforward, backstepping and sliding mode controllers, which deal with a mechanical system with friction that possesses hysteresis nonlinearity due to the micro-sliding friction. The controllers are optimized based on the developed geometric equivalent system.

In the following, Section 2 and Section 3 discuss briefly the theoretical basis of the hysteresis in friction and the geometric nonlinear equivalence. Subsequently, Section 4 presents the derivation of the geometric nonlinear equivalence of a mechanical system with friction element. In Section 5, the derivation of appropriate control structures from the equivalent model and the dynamic simulation of the controlled systems will be discussed. Still in the same section, the effectiveness of the developed control structures will be evaluated and compared with some classical control structures. Finally, some appropriate conclusions are drawn and discussed in Section 6.

### 2 Friction in a nutshell

Friction has been distinguished into two main regimes: the pre-sliding and sliding regimes. When a contacting body is sliding and moving away from a reversal point exceeding a pre-sliding limit – this stage is commonly called as a sliding regime –, the friction force predominantly appears to be a function of velocity. On the contrary, when the velocity of the contacting body is decreased and the motion is reversed, the system is entering a pre-sliding regime, and the frictional effects of the mechanism are predetermined also by a function of displacement. At this stage, the friction behavior depends not only on the velocity, but also on the displacement, where particularly the relation between friction and displacement involves a so-called non-local memory hysteresis.

The key features of this behavior, as also illustrated in Figure 1, are: (i) the friction always follows a certain profile in function of the displacement (referred to as the virgin curve), passing through all reversal points, which have to be memorized, but, (ii) every time a loop of the hysteresis curve is closed, the last reversal point can be forgotten. Al-Bender and Symens, [20], detailed the behavior of the corresponding rolling element friction.

![Figure 1: Rolling friction force behavior: the upper panel shows the displacement input, the lower panel depicts the function of friction force vs. displacement. When a mass subjected on the friction force is displaced, initially the friction force follows an odd function of displacement, the so called virgin curve, \( y(t) \). If the motion is reversed at \( x_m \), the trajectory of the friction force follows the ‘flipped and double stretched’ of \( y(t) \) and the system memorizes the reversal point 1.](image-url)
Despite its complexity, the above described behavior – especially for the case of rolling friction – can be modelled relatively simple by using a parallel connection of Maxwell-slip elements also called Jenkin’s elements (see e.g. [21]). The behavior of a Maxwell-slip element can be represented by two state equations. If the elementary model sticks, which we refer to the presliding regime, it behaves like a linear spring, until it reaches a maximum force it can sustain. Beyond this point the elementary friction force will be equal to the maximum force and at this stage it is referred to the sliding regime friction. When the motion is reversed, the elementary model will stick, and it will behave again like a linear spring until the friction force reaches the maximum force at the opposite direction.

On the contrary, in a case of dry friction, when the motion is entering the sliding regime, the friction force of the elementary model is not equal to a single maximum value, yet the friction force is bounded by a maximum value (static force, \( F_s \)) at the beginning of the sliding motion, and then continues to decrease with increasing sliding velocity toward the Coulomb friction, \( F_C \), which is characterized by state-rate function. Lampaert et al. ([22]) and Al-Bender et al. ([23]) evaluated and formulated the model by merging the two equations for presliding and sliding regimes with a switching function, which is named as the Generalized Maxwell-Slip (GMS). If the elementary model sticks, it behaves like a linear spring, thus the elementary friction force, \( F_i \), can be modelled mathematically as:

\[
\frac{dF_i}{dt} = \kappa_i v \quad \text{(stick)}
\]

with \( \kappa_i \) being the stiffness of each elementary model and \( v \) is the velocity. The elementary model will slip if the friction force of each element reaches the maximum value of the force, \( W_i \), that it can sustain. Beyond this point the elementary friction force is characterized by a state-rate equation:

\[
\frac{dF_i}{dt} = \text{sgn}(v) \cdot C \cdot \left( \alpha_i - \frac{F_i}{s(v)} \right) \quad \text{(slip)}
\]

where \( \alpha_i \) represents the normalized maximum friction force of each element can sustain, and a constant parameter \( C \) implies the rate of the friction force to be attracted to the Stribeck function, \( s(v) \). The higher \( C \), the faster friction force is attracted to the Stribeck function in the sliding regime. The Stribeck function can be represented intuitively by an exponential function, which bounds the friction force by the static friction, \( F_s \), and Coulomb friction, \( F_C \):

\[
s(v) = F_C + (F_s - F_C) \cdot e^{-\frac{|v|}{V_S}}
\]

where \( V_S \) and \( \delta \) are the shaping factors of the Stribeck function, the friction force in the sliding regime always lies between \( F_S \) and \( F_C \). Bögli et al. ([24]) improved the model, by incorporating the formulation in the two regimes to avoid the switching action and the model is referred to the S-GMS (Smoothed Generalized Maxwell-Slip). This approach allows a smooth and continuous numerical calculation.

3 Skeleton Technique

Considering a vibration equation of a single-degree-of-freedom system:

\[
\ddot{y} + 2h_0(A)\dot{y} + \omega_0^2(A)y = 0
\]

where \( y \) is the response signal, \( m \) is the mass of the system, while \( h_0 \) and \( \omega_0 \) are the symmetrical viscous damping and stiffness characteristic of the system, respectively, which depend on the amplitude \( A \). Feldman ([19]) shows that equation (4) can be converted by the Hilbert transform to an analytic signal form:

\[
\hat{\dot{Y}} + 2h_0(A)\hat{\dot{Y}} + \omega_0^2(A)\hat{Y} = 0
\]

where \( \hat{Y} \) is an analytic signal of the response of the system, which can be represented in the form of the combination of slow varying function called envelope, \( A(t) \), and instantaneous phase, \( \phi(t) \):
Taking the derivatives of the analytic signal, \( Y(t) \), and solving the equations of the real and imaginary parts from equation (5), the modal parameters can be written in the expressions of the instantaneous parameters:

\[
\omega_0^2(t) = \omega^2 - \frac{\dot{A}}{A} + \frac{\ddot{A}}{A^2} + \frac{\dot{\omega}}{A\omega}
\]

(7)

\[
h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega}
\]

(8)

where \( \omega \) is the time derivative of the instantaneous phase \( \phi \).

In his paper, Feldman presented the extraction of the envelope, \( A \), and instantaneous frequency, \( f \), by using the Hilbert transform. However, it is reported that the extraction of the parameters will introduce errors when the system is highly damped (see Ruzzene et al., [25]). As an alternative, Staszewski ([26]) proposed to implement the wavelet technique utilizing the complex Morlet mother wavelet function, where subsequently is validated experimentally by Tjahjowidodo et al. ([27]). Due to space limitation, interested readers on the envelope and instantaneous frequency extraction using wavelet transform are suggested to refer to Staszewski ([26]) and Tjahjowidodo et al. ([27]).

4 Geometric Nonlinear Equivalence of a Frictional System

This section discusses the analysis of the dynamic behavior of a mechanical system subjected to frictional forces. Geometric nonlinear equivalence of the mechanical systems, which is represented by a mass block supported by equivalent (nonlinear) spring and damper are evaluated. The motivation of analyzing the equivalent systems by means of the skeleton technique is to understand the dynamic behavior of the system from the perspective of geometric nonlinearity as it is relatively easier than those with material nonlinearity and as a tool to develop an appropriate control structure.

As a simulation, a mass supported by a linear spring and resting on a frictional surface are modeled as illustrated in Figure 2, where the free vibration equation of the frictional system can be written as:

\[
m\ddot{x} + f_{fr}(x, \dot{x}) = F(t)
\]

(9)

where the external force \( F(t) \) is zero, \( x \) is the displacement output of interest, \( m \) is the equivalent mass of the mechanical system and \( f_{fr} \) is the friction force from the interaction between two contact surfaces.

![Figure 2: Schematic drawing of a mechanical vibration system subjected to frictional force, \( f_{fr} \).](image)

For simulation purpose, a set of the system parameters is selected corresponding to equation (1)-(3) and (4.1). It has to be highlighted here that these parameters are chosen arbitrarily but based upon realistic values. Since the values are not based on physical measurements and only qualitative results are examined, no units are assigned to the numerical values of the different variables. For the purpose of performing the dynamic simulation of the corresponding system using the GMS model, 10 elementary models, resulting in 20 (= 2 x 10) parameters of \( \alpha_i \) and \( \kappa_i \) are assigned. The parameters are listed in Table 1 and Table 2, respectively, where parameters \( \alpha_i \) is unitless and \( \kappa_i \) is in stiffness unit, where Figure 3
illustrates the typical friction force, \( f_{fr} \), under harmonic displacement that possesses nonlocal memory hysteresis property.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
m & F_S & F_C & V_S & \delta & C \\
0.01 & 1.0 & 1.0 & 10 & 1 & 60 \\
\hline
\end{array}
\]

Table 1: Parameter set of the simulated system

<table>
<thead>
<tr>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_3 )</th>
<th>( \alpha_4 )</th>
<th>( \alpha_5 )</th>
<th>( \alpha_6 )</th>
<th>( \alpha_7 )</th>
<th>( \alpha_8 )</th>
<th>( \alpha_9 )</th>
<th>( \alpha_{10} )</th>
<th>( \Sigma \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1071</td>
<td>0.0024</td>
<td>0.0826</td>
<td>0.0978</td>
<td>0.1262</td>
<td>0.1407</td>
<td>0.1392</td>
<td>0.1278</td>
<td>0.0883</td>
<td>0.1403</td>
<td>1.0524</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \kappa_1 )</th>
<th>( \kappa_2 )</th>
<th>( \kappa_3 )</th>
<th>( \kappa_4 )</th>
<th>( \kappa_5 )</th>
<th>( \kappa_6 )</th>
<th>( \kappa_7 )</th>
<th>( \kappa_8 )</th>
<th>( \kappa_9 )</th>
<th>( \kappa_{10} )</th>
<th>( \Sigma \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0710</td>
<td>0.0168</td>
<td>0.4157</td>
<td>0.3489</td>
<td>0.3196</td>
<td>0.2526</td>
<td>0.1773</td>
<td>0.1154</td>
<td>0.0566</td>
<td>0.0638</td>
<td>2.8377</td>
</tr>
</tbody>
</table>

Table 2: Elementary parameters of the GMS model

To evaluate the geometric nonlinear equivalence of the corresponding system, some simulations are performed on the Matlab/Simulink environment. In this simulation, the GMS friction model is implemented in an S-function block written in C language to allow fast computation time. Regarding to the study of the FreeVib method, the simulated system is freely oscillating by prescribing a non-zero initial velocity, \( v_0 \). Setting different initial condition will result in a different section of equivalent (nonlinear) stiffness and damping function. Therefore, in order to have complete pictures of the equivalent dynamic parameters, the simulation has to be performed repetitively by using varying initial conditions.

Eight skeleton analyses are performed with different initial velocity, from \( v_0 = 10 \) until \( v_0 = 100 \) and the resulting equivalent modal parameters are presented in Figure 4. Different initial velocity, or equivalently, different energy of the signal will manifest itself at different region in the equivalent dynamic parameters as shown in the figures. In turn, composing all of partial equivalent parameters, complete sets of the dynamic properties are constructed. Fitting the data with an analytical function a combination of exponential function and gamma function is proposed for the restoring force, \( F_k \), while an exponential function is mainly utilized for characterizing the damping function, \( F_d \):

\[
F_k(x) = 0.38 \cdot (1 - e^{-3.2x}) + \frac{0.4}{\Gamma(1.3x^{0.6})} 
\]

\[
F_d(x) = 1.15 \cdot (1 - e^{-0.3(x-0.05)}) 
\]

More detailed discussion on the derivation of the geometric nonlinear equivalence and the analysis of the dynamic performance of the equivalent system is discussed intensively in Tjahjowidodo (2011).

Figure 3: Typical friction as a function of displacement for the system under study
In this section, position control incorporating friction compensation developed based on the geometric nonlinear equivalent system will be discussed. For this purpose, the developed equivalent model is being used as the basis to design the control structures.

5.1 Model-based feedforward compensation

In the first attempt, the models of the frictional system are incorporated in the feedforward loop in combination with a feedback loop. The control scheme is depicted in Figure 5. The feedback loop, which is required to track set-point changes and to suppress unmeasured disturbances, is chosen with proportional gain, $K_p = 100$ and derivative gain, $K_D = 1$. As for the model to be used in the feedforward loop the geometric equivalent system is considered and in order to measure the effectiveness of the model in capturing the dynamic behaviour of the actual system, the compensation performances will be compared and contrasted with no feedforward action and Coulomb model in the feedforward loop. Quantitatively, the root-mean-square (RMS) tracking error is used for different reference input signals to quantify the performance, which is formulated as follows:

$$RMS(y) = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \bar{y})^2}{N}}$$  \hspace{1cm} (12)

Two different reference signals were employed to validate the friction compensation. The first reference position signal is a filtered random signal with a very small stroke in order to emphasize the pre-sliding regime of the friction torque, while the second one is a similar signal but with larger amplitude. The reference signal was generated using a uniform random signal generator with a maximum value of 0.1 for the first reference and 10 for the second signal, which is driven through a low pass, fourth order digital Butterworth filter with 1Hz cut-off frequency.

The tracking error of each friction compensation model for the case of low stroke signal can be seen in Figure 6 (all these results were obtained with the same PID controller parameters.), while those for the high stroke reference input are presented in Figure 7. The RMS values are also presented in the figures.
Based from the simulation results for the low stroke reference input as shown in Figure 6, the geometric nonlinear equivalent system in the feedforward scheme yields a minimum RMS error compared to the other two. On the contrary, the Coulomb model in the feedforward loop exhibits the least performance, since the reference signal has a very small stroke that emphasizes the pre-sliding regime. As the Coulomb model defines the friction force for non-zero velocity, therefore at the vicinity of zero velocity the friction force is only jumping from the positive to negative Coulomb force value and vice versa. In addition, the tracking error of the controlled system with the geometric nonlinear equivalent system exhibits better performance compared to that without feedforward compensation. This demonstrates the ability of the geometric nonlinear equivalence to capture the dynamic behaviour of the frictional system.
For the case of high stroke reference input, as presented in Figure 7, the feedforward compensation with the Coulomb model yields a reasonable performance compared to that with the equivalent model. The reason is due to the large displacement, which is dominated by the sliding regime friction. As presented in Figure 4, at high displacement, the stiffness effect is diminishing, while the system is predominantly characterized by the damping element that resembles to the Coulomb friction function. Nevertheless, considering the performance of the controlled system without feedforward scheme, both models improve the performance of the system.

5.2 Backstepping controller

In this section, the backstepping controller (Khalil, 2002) that is developed based on the equivalent nonlinear system will be presented to compensate for a system with friction element as shown in the equation (9).

Theorem 1: Consider a system $\dot{x} = f(x)$; where $x = 0$ is the equilibrium point and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$ and letting $v: D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $v(0) = 0$; $v(x) > 0$ in $D-\{0\}$.

- if $\dot{v}(x) \leq 0$ in $D$ then the system is stable
- if $\dot{v}(x) < 0$ in $D-\{0\}$ then the system is asymptotically stable

Lemma 1: Consider a system:

$$\dot{y} = f(y) + G(\eta)\epsilon$$

$$\dot{\epsilon} = f_1(y, \epsilon) + G_1(y, \epsilon)u$$

where $y \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}^n$, and $u \in \mathbb{R}^n$, in which $m \geq 2$.
Suppose \( f, f_1, G \) and \( G_1 \) are smooth functions over the domain of interest, \( f \) and \( f_1 \) are vanished at the origin \( (f(0) = f_1(0) = 0) \) and suppose that (5.2) can be stabilized by a smooth feedback \( \varepsilon = \phi(y) \), \( \phi(0) = 0 \), and the Lyapunov function \( V \) that satisfies: \[
\frac{dV}{dy} \left[ f(y) + G(y)\phi(y) \right] < -W(y), \text{ for any } W(y) > 0,
\]
then the system can be stabilized by control variable, \( u \), where:
\[
V_c = V(y) + \frac{1}{2} [\varepsilon - \phi(y)]^T [\varepsilon - \phi(y)] \text{ and } \dot{V}_c = \dot{V}(y) + [\varepsilon - \phi(y)] [\ddot{\varepsilon} - \dot{\phi}(y)].
\]

Considering the mechanical system (4.1) supported by geometric nonlinear elements (4.2) and (4.3), by assuming:
\[
x_{1e} = x_1 - x_d \\
x_{2e} = x_2 - \dot{x}_d
\]
the dynamic of the trajectory tracking of the system can be represented as:
\[
\dot{x}_{1e} = x_{2e} \\
\dot{x}_{2e} = \frac{1}{m} F - \frac{1}{m} (f_d + f_k) - \ddot{x}_d
\]

Subsequently, by defining the Lyapunov function, \( V = \frac{1}{2} x_{1e}^2 + \frac{1}{2} x_{2e}^2 \), the differential form can be represented as:
\[
\dot{V} = x_{1e} \dot{x}_{1e} + x_{2e} \dot{x}_{2e} = x_{1e} x_{2e} + x_{2e} \left[ \frac{1}{m} F - \frac{1}{m} (f_d + f_k) - \ddot{x}_d \right]
\]
In order to ensure the stability of the system, the following control input is selected:
\[
F = m \left[ \frac{1}{m} (f_d + f_k) + \ddot{x}_d - kx_{2e} - x_{1e} \right], \text{ where } k > 0
\]
In such \( \dot{V} = x_{1e} x_{2e} + x_{2e} x_{1e} - kx_{2e}^2 = -kx_{2e}^2 > 0 \)

Figure 8 shows the controlled trajectory tracking result of the frictional system under low stroke reference signal excitation as presented in the previous sub-section. The upper left panel (a) presents the low stroke reference input, the upper right panel (b) shows the trajectory error of the system with backstepping controller, the lower left panel (c) depicts the control input \( F \), while the lower right one (d) shows the contribution of the geometric nonlinear equivalent model on the control input, \( f_d + f_k \). However, it has to be noted that the control parameter, \( k = 5000 \), is not optimized. The parameter is chosen arbitrarily to demonstrate the performance of the model of the equivalent system to compensate for the trajectory error in the frictional system. The tracking result shows a drifted trajectory as illustrated in the trajectory error in Figure 8(b). The equivalent model is shown not to perform satisfactorily in the backstepping controller scheme. Substituting the control input (19) to equation (9), the dynamic equation is becoming:
\[
m\ddot{x} + mk\ddot{x} + mx + \text{resid}(x, \dot{x}) = m\ddot{x}_d + mk\ddot{x}_d + mx_d
\]
where: \( \text{resid}(x, \dot{x}) = f_p(x, \dot{x}) - f_d(\dot{x}) - f_k(x) \)

The dynamic term, represented by \( \text{resid}(x, \dot{x}) \), in equation (20) might result in a residual damping term in addition to the damping term that is linearly proportional with the control gain, \( k \) (the second term in equation 20). Introducing high control gain will results in high damping factor and relative weak spring, as the restoring effect (the third term) is independent to \( k \). Consequently this might lead to a drifted trajectory.

As an alternative, a new stability criterion in term of the Lyapunov function is defined:
\[
V = \frac{1}{2} (x_{1e} + x_{2e})^2 + \frac{1}{2} x_{1e}^2, \text{ where the derivative is presented as:}
\]
\[
\dot{V} = (x_{1e} + x_{2e})(\dot{x}_{1e} + \dot{x}_{2e}) + x_{1e}\dot{x}_{1e} = x_{1e}^2 + 2x_{1e}x_{2e} + (x_{1e} + x_{2e}) \left[ \frac{1}{m} F - \frac{1}{m} (f_d + f_k) - \ddot{x}_d \right]
\]

Selecting \( F = m \left[ \frac{1}{m} (f_d + f_k) + \ddot{x}_d - (k + l)(x_{1e} + x_{2e}) \right] \), with \( k > 0 \), results in
\[
\dot{V} = -x_{1e}^2 - k(x_{1e} + x_{2e})^2,
\]
where the system is ensured to be stable.

![Figure 8](image1.png)

Figure 8: Trajectory tracking application of the system with frictional element using the first backstepping controller

![Figure 9](image2.png)

Figure 9: Trajectory tracking application of the system with frictional element using the second backstepping controller
Figure 9 depicts the controlled trajectory tracking result using the second approach of the backstepping controller for the frictional system under low stroke reference signal excitation. The upper left panel (a) presents the reference input, while the upper right panel (b) shows the trajectory error. Similar to the previous one, the lower left panel (c) depicts the control input \( F \), and the lower right panel (d) shows the contribution of the geometric nonlinear equivalent model on the control input, \( f_d + f_k \). Substituting the control input (19) to equation (9), the dynamic equation is written as:

\[
mix + m(k+1)\dot{x} + m(k+1)x + resid(x, \dot{x}) = m\ddot{x}_d + m(k+1)\dot{x}_d + m(k+1)x_d
\]  

(21)

Compared to the dynamic in the previous backstepping approach (20), the latter one is able to compensate for the residual friction force as the stiffness term (the third term in the left hand side) is also dependent to the controller gain parameter, \( k \). Increasing the parameter will increase the stiffness of the controlled system, which in turn will overcome the drifting problem in the trajectory tracking system as illustrated in Figure 9(b).

### 5.3 Sliding mode controller

Considering a system, written in a state-space form:

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = f(x) + g(x) \cdot u
\]

where \( f(x) \) and \( g(x) \) may be nonlinear functions and \( g(x) \) is a positive definite function, the system is insured to be stable if \( \dot{x}_1 = -ax_1 \), for any \( a > 0 \). Therefore, we define a time varying coordinate, \( s \), where:

\[
s = x_2 + ax_1
\]

(22)

so that if \( s = 0 \), \( \dot{x}_1 = x_2 = -ax_1 \) is stable.

In order to evaluate the stability of this manifold, let us consider the Lyapunov candidate of \( V = \frac{1}{2} s^2 \). In the sense of Lyapunov stability, the derivative of \( V \) should necessarily be negative semi definite:

\[
\dot{V} = \dot{s} s = s \cdot \dot{x}_2 + ax_1 = s \cdot [f(x) + g(x) \cdot u + ax_2] \leq 0
\]

Introducing the variable \( \beta(x) = -[f(x) + ax_2]/g(x) \), the stability of the system is assured if:

\[
\begin{align*}
\dot{s} &= \beta(x) & \text{for } s > 0 \\
\dot{u} &= \beta(x) & \text{for } s = 0 \\
\dot{s} &= \beta(x) & \text{for } s < 0
\end{align*}
\]

For trajectory tracking purpose, considering the mechanical system (9) and assuming the error variables in (15) – (16), a sliding plane \( s \) is defined:

\[
s = \dot{x}_e + ax_{ie}, \text{ and } \dot{s} = \dot{x}_{2e} + ax_{ie} = \ddot{x}_e - \ddot{x}_d + a\dot{x}_{ie}
\]

(23)

Substituting (18) to (23), in order to ensure the stability of the sliding plane equation can be expressed as follows:

\[
\dot{s} = \frac{1}{m} F - \frac{1}{m} (f_d + f_k) - \ddot{x}_d - a\dot{x}_{ie} = 0
\]

Introducing a new variable, \( \beta = f_d + f_k + (\ddot{x}_d - ax_{2e})m \), the control input to assure the stability of the system is selected:

\[
F = \beta - \kappa \cdot \text{sign}(s); \text{ where } \kappa > 0
\]

Thus, \( F = f_d + f_k + (\ddot{x}_d - ax_{2e})m - \kappa \cdot \text{sign}(\dot{x}_e + ax_e) \)

Similar to the previous case, Figure 10 presents the controlled trajectory tracking result of the frictional system under low stroke reference signal excitation using sliding mode controller. The upper left panel (a)
presents the low stroke reference input, the upper right panel (b) shows the trajectory error of the system, the lower left panel (c) depicts the control input $F$, while the lower right one (d) shows the contribution of the geometric nonlinear equivalent model on the control input, $f_d + f_k$. The parameters of the controller, $a = 100$ and $\kappa = 10$ are also not optimized, since the objective is to demonstrate the ability of the equivalent system to capture the dynamic properties of the frictional system. The compensated system exhibits satisfactory trajectory performance as shown in panel (b) of Figure 10 with RMS = $2.08 \times 10^{-4}$, while as shown in the last two panels (c and d), the model of the equivalent system contributes also significantly to the control input to the system. In particular, no drifting in the trajectory occurs as appears in the first approach of the backstepping controller. This is due to the compensation of the uncertainty represented in the signum function.

Figure 10: Trajectory tracking application of the system with frictional element using sliding mode controller

6 Conclusion

A geometric nonlinear equivalence of a nonlocal memory hysteresis friction can be derived from the dynamic perspective by means of the skeleton method. The equivalent spring shows a very high stiffness at low displacement, which subsequently drops at high displacement. The equivalent damper behaves linearly at low sliding velocity, which is saturating at high velocity to a damping force corresponds to the break-away force.

This paper, in particular, has evaluated and presented the effectiveness of the equivalent system for compensation scheme using three distinct controller strategies, namely the feedforward, backstepping and sliding mode. The equivalent system is shown to effectively compensate for the frictional effect by implementing it in the feedforward loop. The results show a consistent performance in low stroke application, where the friction is predominantly characterized by the presliding regime, and also in high stroke application. In contrast, a Coulomb model, disregarding the stiffness effect in the presliding regime, shows comparable performance only at high stroke application.

When the equivalent system is implemented in the backstepping controller, the performance depends on the selection of the control input form. For the case when the control input is not carefully selected, the compensated system might result in a weak (virtual) restoring effect. Consequently, this will lead to a drifted trajectory tracking because of the presence of the residual damping term. However, by properly
selecting the control input form, it is shows that the equivalent system is able to compensate for the frictional effect using the backstepping controller. While in the last case, the sliding mode controller constructed using the equivalent system exhibits a satisfactory performance, even for relatively low uncertainty parameter, $\kappa$.

**References**


