Experimental chaotic quantification in bistable vortex induced vibration systems

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ABSTRACT

The study of energy harvesting by means of vortex induced vibration systems has been initiated a few years ago and it is considered to be potential as a low water current energy source. The energy harvester is realized by exposing an elastically supported blunt structure under water flow. However, it is realized that the system will only perform at a limited operating range (water flow) that is attributed to the resonance phenomenon that occurs only at a frequency that corresponds to the fluid flow. An introduction of nonlinear elements seems to be a prominent solution to overcome the problem. Among many nonlinear elements, a bistable spring is known to be able to improve the harvested power by a vortex induced vibrations (VIV) based energy converter at the low velocity water flows. However, it is also observed that chaotic vibrations will occur at different operating ranges that will erratically diminish the harvested power and cause a difficulty in controlling the system that is due to the unpredictability in motions of the VIV structure. In order to design a bistable VIV energy converter with improved harvested power and minimum negative effect of chaotic vibrations, the bifurcation map of the system for varying governing parameters is highly on demand.

In this study, chaotic vibrations of a VIV energy converter enhanced by a bistable stiffness element are quantified in a wide range of the governing parameters, i.e. damping and bistable gap. Chaotic vibrations of the bistable VIV energy converter are simulated by utilization of a wake oscillator model and quantified based on the calculation of the Lyapunov exponent. Ultimately, a series of experiments of the system in a water tunnel, facilitated by a computer-based force-feedback testing platform, is carried out to validate the existence of chaotic responses. The main challenge in dealing with experimental data is in distinguishing chaotic response from noise-contaminated periodic responses as noise will smear out the regularity of periodic responses. For this purpose, a surrogate data test is used in order to check the hypotheses for the presence of chaotic behavior. The analyses from the experimental results support the hypothesis from simulation that chaotic response is likely occur on the real system.

1. Introduction

Vortex induced vibrations of elastically constrained structures in water flows have recently received intensive attentions as a promising and potential alternative among many available sources of renewable energy. The first vortex-induced-vibrations based energy converter trademarked as VIVACE was introduced in [1]. The VIVACE energy converter consists of a cylinder submerged...
perpendicularly into a flow. The cylinder is elastically supported by a linear spring and its motion is constrained along the cross-flow direction, thus allowing it to move in one degree of freedom. When the water flow passes the cylinder, it will induce periodic drag and lift forces imposing along the surface of the cylinder that is due to the occurrence of vortices alternately shedding on two sides of the wake region behind the cylinder. Under the effect of lift forces, the cylinder will vibrate in the cross-flow direction since it is constrained by the spring in this direction. These vibrations of the cylinder are known as VIV (vortex induced vibrations), whose elaborate characteristics are widely reviewed in ([12–4]). A transmission mechanism is utilized to connect the cylinder to a generator and convert the VIV into rotations of the shaft in the generator. In this way, usable energy, i.e. electrical energy, is produced from kinetic energy of water flows.

A critical parameter in designing a VIV energy converter is its natural frequency, which essentially is determined by the spring stiffness and the effective mass of the oscillating structure. In order to maximize the harvested power, the natural frequency of the system must coincide with the exciting frequency of lift forces, which is determined by velocity of the water flow. This resonance requirement will allow the VIV energy converter to be designed effectively for specific water flow conditions. This outstanding merit makes this approach for energy harvesting potential and promising. Unfortunately, another issue arises for a VIV energy converter operating in practical scenario, where water flow velocity in nature typically fluctuates severely. When the flow velocity deviates far from the resonance frequency, the effectiveness of the energy converter will drop significantly. Since this adversity was revealed, there were several theoretical and experimental studies (e.g. [5–10]), which focused on characterizing and optimizing the designing parameters of a VIV energy converter through maximizing the resonance range and increasing vibrating amplitude. The parameters include mass, damping, stiffness and surface roughness of the vibrating structure.

Among those studies, the significantly positive effect from the approach of appending a nonlinear stiffness element to the system in order to broaden the resonance range is discussed. Experimental studies in [11] and [12] have shown that a VIV energy converter with a hardening stiffness element has widened the resonance range toward the high operating working frequencies, i.e. high speed water flows. In contrast, a bistable stiffness element is capable to improve the harvested power at low operating frequencies ([13]). A numerical analysis, which will be discussed in details later in Section 2, also shows that bistable stiffness can significantly improve harvested power from a VIV energy converter at low velocity water flows. However, when parameters vary, including the water flow velocity, the harvested power might drop drastically due to the chaotic behavior that might manifest itself in the vibration response. Chaotic signal is defined as a non-periodic, unpredictable and seemingly noisy vibration [14]. It might manifest themselves in any nonlinear mechanical systems and commonly it appears in those with severe nonlinearities, such as mechanical systems with backlash ([15–17]) or those with hysteresis ([18–20]).

Chaotic vibrations are evidently undesired for the operation of a VIV energy converter because of the low harvested power and the unpredictability in motions that will result in difficulties in controlling the system. For this reason, the feasibility in application of bistable stiffness in a VIV energy converter to improve its performance requires a design for the converter that can avoid chaotic vibrations from its operating range. The prerequisite to design such a converter is the knowledge of chaotic response dependency to the governing parameters of the system.

Zhao et al. [21] studied chaotic responses of a VIV structure in laboratory scale. Initially, they measured the state variables of the system, i.e. the output displacement and lift force, on a testing apparatus. Subsequently, they confirmed the existence of chaotic vibration by prescribing the recorded displacement on a position-controlled-testing-platform and measured the lift forces. From the observation, they concluded the existence of chaotic vibrations when large discrepancies occur between the two measured lift forces. Perdikaris et al. [22] studied vibrations on a VIV system in a constant flow for varying amplitudes and fixed frequency. They observed that chaotic vibration manifests itself in a case of moderate amplitudes and confirmed the chaotic phenomenon by analyzing the frequency spectrum of the lift force, where no dominant frequency appears in the spectrum. Blackburn and Henderson [23] used the 2D numerical simulation to investigate the lock-in behavior of the cylinder excited by a constant flow. They confirmed chaotic vibrations at a certain frequency ratio by observing non-periodic vibrations and the auto-spectrum of the cross-flow vibrations, where the spectral peaks do not match to the natural frequency and the Strouhal frequency.

In this paper, chaotic vibrations of a bistable VIV energy converter will be quantified in a wide range of the two governing parameters: damping and bistable gap, which are figured out by a dimensional analysis on the system. The quantification will be mainly based on the largest Lyapunov exponent of the time series displacement of the cylinder, which can be considered as the footprint of chaos if it is positive and its magnitude measures the degree of chaos in the system ([24]). Therefore, its dependency on damping and bistable gap will serve as a guideline to design a bistable VIV energy converter.

The calculation of the largest Lyapunov exponent relies on the nonlinear time series analysis procedure, which is systematically presented in [25–27] and was successfully applied to analyze the chaotic responses of some mechanical systems in [15] and [28]. To evaluate, a time delay phase space is reconstructed from the time series data to calculate the largest Lyapunov exponent. In addition, dealing with experimental data requires a procedure to distinguish the presence of chaotic behavior in the non-linear system and of the linearly correlated noise and the surrogate data testing will be carried out to perform this task. This approach offers a more consistent chaotic quantifier and is capable of assessing the chaotic phenomenon from a single data set.

The paper is organized as follows: In Section 2, the wake oscillator model that is utilized to simulate the time series displacement of the structure will be presented. The expression of a bistable stiffness element applied in the VIV energy converter and the effect of bistable stiffness on performance of a VIV energy converter will also be discussed in this section. Elaborative discussion on the procedure of reconstructing the time delay phase space and calculating the largest Lyapunov exponent as well as results from this procedure can be found in Section 3. While Section 3 mainly discusses the analysis on simulated data, Section 4 presents the experimental setup of a bistable VIV system in a real water tunnel, data analysis and implementation the surrogate data testing to verify the presence of chaotic responses. Some key conclusions from this study are drawn in Section 5.
2. Background

This section concentrates on the numerical simulation of vortex induced vibration systems utilizing the wake oscillator model. The discussion comprises the dynamic modeling of the system with bistable springs, dimensional analysis of the nonlinear system, a brief comparison on the performance of the VIV system with linear and bistable spring and a particular case when the nonlinear system results in chaotic behavior.

2.1. VIV modeling by wake oscillator model

For simulation purposes, the VIV system is modeled as shown in Fig. 1. The system consists of a cylinder, as the oscillating blunt object, with an effective mass of $m$, diameter of $D$ and length of $L$. Referring to Fig. 1, the cylinder length is oriented along the $z$-direction and is submerged in a water stream along $x$-direction with a constant flow velocity of $U$. The cylinder is supported by a linear spring with a stiffness constant of $k$, and is constrained in such it can only vibrate in $y$-direction. The effective mass and the stiffness value determine the natural frequency of the system; thus, will also define the resonance phenomenon that corresponds to the vortex shedding frequency from the water flow. The structural damping, $c$, that comprises the damping from the mechanical structure and that from the PTO (power take-off) box, will define the resonance range (please note that the harvested power, from analytical point of view, will be considered to be proportional to the damping from the PTO). In particular, Khalak and Williamson ([29–31]) reported the mass ratio, $m^*$, which represents the ratio between the effective mass and the fluid mass displaced by the cylinder, will also determine the resonance range.

To dynamically simulate the system, the wake oscillator model, which was proposed in [32], modified and validated in [33], is utilized. In this model, the dynamic equation of the VIV structure (Eq. 1) is coupled with the nonlinear wake oscillator equation (Eq. 2) to simulate the waking effect of the vortex shedding in the corresponding vortex-induced-vibration system.

\[ m\dddot{Y} + r\dot{Y} + h\dot{Y} = S(q) \]  
\[ \ddot{q} + \epsilon (2\pi St (U/D)) (1 - \beta q^2 + \lambda q^4)q + [2\pi St (U/D)]^2 q = F(\dddot{Y}) \]  

where:

1. $m$: sum effective mass and fluid added – mass
2. $r$: sum of structural damping and fluid added – damping
3. $h$: spring stiffness
4. $Y$: displacement of cylinder in cross-flow direction
5. $S(q)$: vortex shedding force
6. $q$: wake variable
7. $St$: Strouhal number
8. $U$: water flow velocity
9. $D$: diameter of cylinder
10. $F(\dddot{Y})$: effect from vibrations of the structure on the wake
11. $\epsilon$, $\beta$, $\lambda$: coefficients of nonlinearity (refer to [32] and [33] for detailed explanations)
2.2. Bistable stiffness modeling and dimensional analysis

A bistable stiffness element has become a long-history research topic due to its unique and sophisticated dynamical characteristics. When a bistable stiffness element is applied an oscillator, it will manifest itself as a two-well potential energy system as illustrated in Fig. 2. From the perspective of nonlinear dynamics, this system has two stable fixed points and one unstable fixed point at the origin since it is never stabilized at this position. When the system is excited with the low excitation energy, it might vibrate within one well of potential energy (1), depending on the initial condition. This mode is referred to as the intra-well vibrating mode, since the system only vibrates within one potential well. When the energy of excitation is sufficiently large to drive the system to overcome the unstable point, the system will vibrate along trajectory (2), which is referred to as the inter-well vibrating mode. Apart from the intra-well and inter-well vibrating modes, the system can also vibrate in a more complex manner, e.g. jumping back and forth between the two potential wells or the two vibrating modes, where in extreme cases will lead to chaotic responses. Effects of bistable stiffness on improving the performance of ambient-vibration-based miniaturized energy harvesters have been intensively studied in some literature (see [34–36]). However, based on authors’ knowledge, it is hardly any evidence in literature the discussion of a bistable stiffness effects on a VIV system.

The left panel in Fig. 3 illustrates the schematic diagram of a mechanical system with the spring components that results in a bistable stiffness characteristic. Unlike a linear system with only the vertical springs \( k_1 \), where the stable point is located at the origin, the appearance of the two oblique springs \( k_2 \) produces an extra nonlinear restoring force component apart from the linear component in the \( y \)-direction and enables the system to rest at either the upper or lower stable points.

By decomposing the restoring forces from the two oblique springs into \( y \)-direction, the equivalent restoring force, \( F_k \), of the system in this direction can be derived and expressed as a smooth nonlinear function that contains a negative stiffness regime in the vicinity of the origin (detailed derivation can be found in [13]).

In this paper, for the analysis that will be carried out in non-dimensional form, the equivalent restoring force function is simplified to a piece-wise function (Eq. 3) that conserves the two vital parameters of a bistable stiffness characteristic: equivalent stiffness \( k_0 \) and bistable gap \( \varepsilon \) as illustrated in the right panel of Fig. 3. By substitution of this function for the third term on the left hand side in Eq. 1, the displacement of the cylinder in a bistable VIV energy converter can be simulated numerically.

\[
F_k (y) = k_0 y - k_0 \varepsilon \text{sign}(y) 
\]  

(3)

From dynamics perspective, the fundamental parameters of a bistable VIV system are the effective mass, \( m \), structural damping, \( c \), equivalent stiffness, \( k_0 \), and bistable gap, \( \varepsilon \). A dimensional analysis is performed to minimize the number of the governing parameters for the parametric quantification of the system. The dynamic equation of a bistable VIV energy converter can be expressed as follows:

![Fig. 2. Two-well potential energy characteristic of a system with bistable stiffness element.](image)

![Fig. 3. A mechanical system with bistable stiffness characteristic.](image)
The system turns out to be governed by two main parameters $\alpha$ and $\gamma$, that is represented in the wake oscillator model. Some non-dimensional parameters are introduced, namely $\epsilon$, where:

$$\epsilon = \frac{\xi}{2\sqrt{\omega_0 m}}.$$  

By substituting these quantities to Eq. (4), its dimensionless form can be obtained as follow:

$$p'' + 2\xi p' + F^*(p) = \gamma F\left(\frac{f}{\omega_0}\right)$$  

where: $F^*(p) = p - \text{sign}(p)$, $\gamma = \frac{1}{\sqrt{\omega_0 m}}$, $p' = \frac{dp}{dt}$ and $p'' = \frac{d^2p}{dt^2}$. The system turns out to be governed by two main parameters $\xi$ and $\gamma$. It should be noted that $\gamma$ depends on $k_0$ and $m$. Since $k_0$ and $m$ determine the natural frequency of the system, $\omega$ (or $\alpha$ in its dimensionless form), will be varied in order to investigate the effect of different $\gamma$.

Typical parameters of a VIV system presented in [31], i.e. $m^* = 10.1$ and $\zeta_0 = 0.00129$, will be utilized in this paper. The natural frequency of the system in water is selected to be $f_{\text{water}} = 0.7$ Hz and the equivalent stiffness, $k_0^*$, is calculated accordingly based on the preselected values. The diameter of the cylinder is chosen to be $D = 0.05$ m to meet the dimension of the water tunnel utilized in the experiment. All these parameters are applied to the coupling equations to simulate the displacement of the cylinder at different water flow velocity, $U$.

### 2.3. Effect of bistable stiffness on a VIV energy converter

Using the wake oscillator model, the harvested power can be estimated. The average vibrational power, $P^*$, is calculated based on the velocity of the vibrating structure, $\dot{y}$. Fig. 4 shows the average vibrational power versus water flow velocity, $U$, from the bistable system with different bistable gap values and the linear system with the equivalent stiffness. It can be observed that the bistable spring exhibits a minor effect on the system with a small bistable gap ($\alpha = 0.02$). When the bistable gap is increased ($\alpha = 0.2$), the harvested power is strikingly improved at low speed water flows due to large amplitude vibrations in the inter-well vibrating mode. However, the power suddenly drops when water flow velocity is increased (beyond 0.175 m/s water velocity). From our observation, it is suspected that the drop in power is attributed to the chaotic vibrations that occur on corresponding water flow velocities.

The occurrence of chaotic vibrations in a bistable VIV energy harvested also depends on the damping imposed to the system. Fig. 5 shows the vibrating response of the system when the damping value, $\zeta$, is changed. In the upper panel of the figure (a), the response of the system is chaotic at a low value of damping, $\zeta = 0.00129$. However, when the damping is increased ($\zeta = 0.0645$), the chaotic effect diminishes and the system vibrates periodically (see Fig. 5b).

Evidently, the bistable stiffness can only improve the harvested power when it is configured by a sufficiently large bistable gap. Unfortunately, increasing the bistable gap apparently is suspected to result in chaotic vibrations. Therefore, the negative effect caused by chaotic vibrations must be taken into account when one wants to apply bistable stiffness to improve harvested power of a VIV energy converter. Besides, imposing a high amount of damping to the system seems to be able to control the chaotic vibrations of the bistable VIV energy converter. To confirm these observations, hereafter, the study will focus on parametrically quantifying chaotic vibrations of a bistable VIV energy converter to map the occurrence of chaotic response as a function of the governing parameters $\alpha$ and $\zeta$.

### 3. Parametric quantification of chaotic vibrations

At this stage, however, it is not certain yet that the suspected chaotic responses are, indeed, chaotic. Therefore, in this section, first, we will discuss some theoretical analysis to quantify the chaotic degree on a response. Lyapunov exponent, which is considered as a credible tool to quantify the chaotic degree in a time series data, will be discussed including the detailed procedure for estimation the quantifier. In the last part of this section, the Lyapunov exponent map of the bistable VIV energy converter for varying...
damping, ζ, and bistable gap, α, will be presented and analyzed in detail.

3.1. Chaotic quantification by Lyapunov exponent

In nonlinear dynamics studies, the dynamical state of a system in time can be presented in phase space. The phase space illustrates an orbit or trajectory of the state variable when the system evolves in time. For a dissipative dynamic system, the behavior will reach a steady state after a transient period; therefore and the trajectory on the phase space will evolve within a bounded region and converge into a point that is referred to as an attractor. For a limit cycle where the vibration is periodic, the phase space will form a closed trajectory. On the contrary, for a chaotic response, the trajectory is non-periodic and it is hardly reproducible. In particular, when a chaotic system is excited with two different sets of nearby initial condition, the trajectories on the phase space will split up exponentially. This leads to the basic concept of Lyapunov exponent to quantify the chaotic degree of a system.

If \( d(t_0) \) represents the distance between the two state points on a phase space that correspond to the two nearby initial conditions of the system, after a certain instance, \( t \), the distance between the two state points, \( d(t) \), can be expressed as:

\[
d(t) = d(t_0) e^{\lambda t}
\]

where \( \lambda \), the Lyapunov exponent, represents the footprint of chaotic vibrations if it is positive. The magnitude of the Lyapunov exponent indicates the diverging speed of the nearby trajectories. Therefore, it can be utilized to measure the chaotic degree of the system.

It has to be noted, however, that the analysis of Lyapunov exponent is barely practical to be performed straightforwardly, i.e. exciting the system with two different sets of initial condition and observe the trajectory separation on the phase plane. As an alternative, Stefanski's method ([37–39]) is proposed based on a synchronization approach. This method, which was successfully performed for many applications, including a mechanical system with impacts and a time delay system, is relied on the synchronization condition of two coupled identical dynamical systems, whose dynamics equations are known. In a case where the dynamics equations are unknown, this method can also be performed based on the two coupled discrete maps constructed from time evolution of the system state. However, in this study, the largest Lyapunov exponent will be evaluated on experimental data, where the measurable state variables are limited. Therefore, it is more suitable to evaluate the largest Lyapunov exponent based on a single time series data, i.e. in this study we use the time series displacement of the VIV. The fundamental procedure for nonlinear...
time series analysis is discussed in [25–27]. To elaborate, from the single time series data, a time delay phase space, where the Lyapunov spectrum of the system in the original phase space can be conserved, will be reconstructed. Then, the Lyapunov exponent is evaluated from the state points that are created by embedding the time series data to the reconstructed phase space.

3.2. Reconstruction of time delay phase space

In this study, the analysis is carried out on the time series data of the cylinder VIV displacement, $y(t)$. When this time series is embedded to the time delay phase space, the state of the system at time $t$ is represented in the following state vector:

$$y(t) = [y(t), y(t + t_d), \ldots , y(t + (d - 1)t_d)]$$

where $t_d$ is the time delay and $d$ is the embedding dimension of the reconstructed phase space.

3.2.1. Selection of time delay

The phase space is reconstructed from the time series data, $y(t)$, with time delay $t_d$. In order to analyze the chaotic behavior on the response, the time delay has to be properly selected. The selection is based upon the minimization of the correlation in each dimension of the phase space. Intuitively, auto-correlation technique will be able to serve for this task, however, it is suggested in [26], the average mutual information (AMI) is more appropriate when one deals with nonlinear system. The formulation of AMI for time series $S(t)$ is presented as follows:

$$I(t, t + \tau) = \sum_{S(t), S(t+\tau)} P(S(t), S(t + \tau)) \log_{2} \frac{P(S(t), S(t + \tau))}{P(S(t))P(S(t + \tau))}$$

where $P(A,B)$ is the joint probability density for set $A$ and $B$, and $P(A)$ is the individual probability density for set $A$. Fig. 6 illustrates the AMI of the time series displacement of the cylinder at $U=0.175$ m/s, $\zeta=0.00129$ and $\alpha=0.25$. It is suggested from the figure that the time delay, $t_d$, is chosen to be 0.369 s, as the AMI reaches its first local minimum at the corresponding time instance.

3.2.2. Selection of embedding dimension

After acquiring the time delay, the embedding dimension is selected to complete the phase space reconstruction. The embedding dimension, $d$, determines the number of elements in a state vector, or the dimensionality of the reconstructed phase space. The basic idea for selection of the embedding dimension is that it must be sufficiently large so that the attractor revealed in the reconstructed phase space is totally unfolded. In this paper, the method proposed in [40] is applied to select the embedding dimension. To elaborate the method, $y(d)$ and $y(d+1)$ are defined as the state vectors in the reconstructed phase space with the embedding dimension $d$ and $d+1$, respectively. The Euclidian distance between each point and its nearest neighbor in the embedding phase space is calculated. Then, the ratio of this distance of the point $y(d+1)$ to one of the point $y(d)$ is defined as the embedding error when the embedding dimension is increased from $d$ to $d+1$. The quantity $E(d)$ that measures the mean value of all the embedding errors of all the reference vector points is calculated. Finally, the selection of the embedding dimension is based on the ratio $E_1(d)$ defined as follows:

$$E_1(d) = E(d+1)/E(d)$$

$E_1(d)$ indicates the change of $E(d)$ when the embedding dimension is increased from $d$ to $d+1$.

Dimension $d_0+1$ is selected as the embedding dimension if $E_1(d)$ starts to saturate at a dimension $d_0$. Fig. 7 illustrates the ratio $E_1(d)$ as a function of the embedding dimension from the time series displacement of the VIV at $U=0.175$ m/s, $\zeta=0.00129$ and $\alpha=0.25$. This ratio saturates when the embedding dimension is larger than 3. Therefore, the embedding dimension $d=4$ is selected in this case.
3.3. Calculation of the Lyapunov exponent

After the time delay and the embedding dimension are evaluated, the phase space is subsequently reconstructed to allow for the Lyapunov exponent calculation. The calculation relies on the direct method of the largest Lyapunov exponent estimation reviewed in [27]. The calculation is started by specifying a state point, $y(t)$, at a specific time instance $t$, and afterwards the Euclidian distance between this point and its nearest neighbor is calculated. The distances between these two points are repeatedly evaluated, for every period of $\Delta t$, while the points evolve in phase space. At the same time, the ratio between the evolving distance values and its original distance are recorded. Subsequently, the averaged ratio, which later is referred to as a prediction error is plotted against the evolution time, $\Delta t$, on logarithmic scale.

Intuitively, the plot represents the growth of the phase space trajectory and if it is exponentially growing, a linear function against the evolution time will be observed on the plot. In other word, a positive slope on the plot will indicate the exponential growth in the distance of nearby trajectories, where the slope estimates the largest Lyapunov exponent that represents the chaotic degree of the time series.

Fig. 8 illustrates the prediction error as a function of $\Delta t$ from the time series displacement of the cylinder at $U=0.175$ m/s, $\zeta=0.00129$ and $\alpha=0.25$. In this case, the largest Lyapunov exponent, $\lambda$, is estimated to be equal to 0.7315 bit/s.

The displacement of the cylinder is simulated for a wide range of governing parameters, i.e. damping and bistable gap, and the largest Lyapunov exponent, $\lambda$, is calculated correspondingly. Fig. 9 shows the largest Lyapunov exponent as a function of $\zeta$ and $\alpha$ for a specific water flow velocity of 0.175 m/s, which corresponds to the estimated resonant velocity calculated from the vortex shedding frequency. The contour plot constitutes of 4941 data points, where each point was evaluated from 5000 s simulated data point with 0.001 s sampling time (but only the last 2000 s was used to calculate the largest Lyapunov exponent to ensure the exclusion of the transient response).

It can be observed that the chaotic degree reduces and subsequently diminishes when the damping imposed to the system is increased. For the values of damping which are larger than 0.0774, chaotic vibrations of the system become trivial. From the contour plot, it can also be seen that when the damping increases, the chaotic region is more likely to occur at small bistable gaps until it diminishes at high damping value. On the other hand, at low bistable gaps, the system tends to result in regular responses. This can be easily understood as the system resembles a linear system. As the bistable gap increases, the chaotic response has higher chance

Fig. 8. Prediction error as a function of $\Delta t$ from the time series displacement of the cylinder at $U=0.175$ m/s, $\zeta=0.00129$ and $\alpha=0.25$. 

\[ \lambda = 0.7315 \text{ (bit/s)} \]
to occur before the largest Lyapunov exponent will gradually decrease as the vibrational energy is not sufficient to develop inter-well vibration mode.

Fig. 10 depicts the bifurcation diagram of the corresponding system for different damping and bistable gap values. The diagrams are simulated as a function of water flow velocity, $U$, where each run were simulated for 20,000 s at 0.001 s sampling time, but only the last 10,000 s was used to evaluate the bifurcation in order to ensure the exclusion of the transient part. The regions with chaotic vibration are indicated by infinite numbers of amplitude on the bifurcation diagram..

In general, at low water flow velocity, the system behaves regularly. In particular, at very low velocity, the system initially stays stationary as no resonance occurs. The system starts to vibrate periodically with increasing amplitude when the water flow velocity is increased due to the resonance phenomenon. Until at a certain flow velocity, the system starts to vibrate chaotically. However, when the water flow velocity is further increased, the system behaves regularly again before the vibrations are totally suppressed at high velocity water flows. For example, in the case of $\zeta=0.0645$, the chaotic response starts at the water flow velocity of 0.14 m/s and it behaves completely regular again beyond the velocity of 0.206 m/s (even though, it is noted that within that velocity range there also exist some regions that exhibit regular responses, e.g. between 0.18 m/s and 0.196 m/s). The abrupt transitions (sudden jumps on the amplitudes) between periodic and chaotic vibrations in the sequence of bifurcations are attributed to the discontinuity of the proposed spring force in the system.

From the four figures in Fig. 10(a), as also concluded from the contour plot in Fig. 9, we can easily see that a higher damping value is capable of diminishing chaotic vibrations on the system. On the other hand, the effect of the bistable gap can be studied from Fig. 10(b). When the bistable gap increases, the chaotic regions extend. This concludes that chaotic vibration is likely to occur on the system with a larger bistable gap. However, as indicated in Fig. 9, when the bistable gap continues to increase, chaotic vibration is less likely to occur, due to insufficient vibrational energy to generate inter-well vibration.

Fig. 11 shows the Poincaré map, which illustrates the intersection of an orbit in the phase space of the dynamical system with a certain lower-dimensional subspace. The observations from the plot also suggest the same tendency as shown in the contour plot and the bifurcation diagrams. The maps are generated by simulating the system in a much longer period of 200,000 s after exclusion of the transient response, in order to have sufficient intersection points in the phase space..

The left panels of the figure, Fig. 11(a), illustrate the evolution of the Poincaré map of the system response for increasing damping values. From the figures, it can be seen that the system results in a chaotic response when $\zeta=0.00129$ as the Poincaré map exhibits a fractal structure. In line with the contour map in Fig. 9, a higher damping value will suppress the chaotic behavior as the map for $\zeta=0.0774$ comprises only a finite number of points, instead of a fractal structure formation. On the other hand, the periodic vibration turns to be chaotic when the bistable gap increases and turns to be periodic again when the bistable gap exceeds a certain value (see the right panels (b)).

In practice, high damping value to suppress chaotic behavior on the VIV energy generator can be attained from the damping of the PTO box, while designing the converter with either a small or very large bistable gap can also be useful in avoiding chaotic vibrations. However, the bistable system with a very large bistable gap tends to vibrate in the intra-well mode dominantly; thus the technical merit of enhancing the harvested power through the introduction of bistable spring will not be achieved.

4. Experimental validation

To verify the existence of chaotic responses on the bistable VIV system, a series of experiment is carried out on a dedicated testing platform.
4.1. Computer-based force feedback testing platform and experimental facilities

A computer based testing platform to facilitate experiments of VIV structure with varying governing parameters is designed and
developed. The system is similar to that presented in \cite{[41–43]}. In this testing platform, the structure that is immersed in a water tunnel is connected to a load-cell to measure forces exerted by the generated vortices. Subsequently, the measured forces are feedback to a controller to determine the lift forces and the instant displacement of the structure. The calculation is based on the governing equation (Eq. 10) that comprises the simulated parameters, i.e. effective mass, damping value and spring characteristics. The

Fig. 11. Poincaré maps recording the displacement of the cylinder when its velocity passes a certain value.

Fig. 12. Computer-based force-feedback testing platform.
calculated displacement is sent to an actuator in the testing platform to drive the VIV structure. The concept of the testing platform is shown illustratively in Fig. 12.

The main advantage of this testing platform is that the physical parameters of the system, e.g. the effective mass, damping and spring characteristics can be virtually, yet easily, imposed to the system. Particularly, the bistable restoring force function (Eq. 3) can be easily prescribed into the system through the controller to apply various bistable springs, without having a need to physically alter it.

\[ m\dot{y} + c\dot{y} + ky = F_{\text{ip}}(t) \]  

The water tunnel at the Maritime Research Center, Nanyang Technological University, facilitated the experiments. The facility is able to deliver water flow velocity from 0.02–0.7 m/s which is conditioned to the uniformity of within 1.5% across the testing section and the turbulence intensity of below 2%. The dimension of the water tunnel is 45×31×500 cm (H×W×L), while the VIV structure, in a form of cylinder, has a length of 23.5 cm and the diameter of 5 cm. A set of physical parameters of \( m^* = 10.1 \) and \( \zeta_0 = 0.00129 \) and \( f_n,_{\text{water}} = 0.7 \) Hz, which were also utilized in Section 3, are applied in the experiment. The time series displacement data of the cylinder is recorded through an 8192 pulse/rev encoder installed in the servo motor, which is connected to the belt-drive actuator that drives the cylinder. A set of parameters as listed in Table 1 are prescribed to the system to investigate different effects of governing parameters. In order to assure the reliability of the experimental data, \( 3 \times 10^6 \) data points were recorded for every 0.002 s for each set of data after the response was completely in steady-state condition, resulting in almost two hours data length.

### 4.2. Surrogate data testing

Fig. 13 illustrates the displacement data of the cylinder in the VIV system from cases 7 and 8. At a glance, one may see a regular response in Case 7 and (more) irregular behavior in Case 8. Even though regular behavior is observed in Case 7; however, it is observed slight irregularity that is potentially attributed to the random noise. Therefore, in order to avoid bias assumption in distinguishing periodic responses from chaotic ones, a surrogate data testing is performed to all sets of experimental data.

**Surrogate data testing**, which was studied in [44] and systematically reviewed in [45], is an effective stochastic tool that can be utilized to figure out whether the irregularity in a time series data is induced by noise (linearly stochastic process) or by the inherent nonlinearity of the system. In this method, the surrogate data is generated using a linear process that is consistent with a specified process defined in a null hypothesis. The generated surrogate data sets are defined to mimic only the linear properties of the original data. Subsequently, a discriminating statistic is calculated for both the original data and the surrogate data. Stochastically, if the value of the original data passes the hypothesis test, the irregularity on the signal is confirmed to be attributed to linear stochastic process, otherwise it is confirmed as a product of nonlinear processes. In this paper, the surrogate data is produced by the Fourier

---

**Table 1**

<table>
<thead>
<tr>
<th>Case no.</th>
<th>( \zeta )</th>
<th>( \alpha )</th>
<th>( U ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \zeta_0 )</td>
<td>0.1</td>
<td>0.13</td>
</tr>
<tr>
<td>2</td>
<td>( \zeta_0 )</td>
<td>0.1</td>
<td>0.18</td>
</tr>
<tr>
<td>3</td>
<td>( \zeta_0 )</td>
<td>0.1</td>
<td>0.20</td>
</tr>
<tr>
<td>4</td>
<td>( \zeta_0 )</td>
<td>0.2</td>
<td>0.13</td>
</tr>
<tr>
<td>5</td>
<td>( \zeta_0 )</td>
<td>0.2</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
<td>( \zeta_0 )</td>
<td>0.2</td>
<td>0.20</td>
</tr>
<tr>
<td>7</td>
<td>20( \zeta_0 )</td>
<td>0.1</td>
<td>0.13</td>
</tr>
<tr>
<td>8</td>
<td>20( \zeta_0 )</td>
<td>0.1</td>
<td>0.18</td>
</tr>
<tr>
<td>9</td>
<td>20( \zeta_0 )</td>
<td>0.1</td>
<td>0.20</td>
</tr>
</tbody>
</table>

---

Fig. 13. Time series displacement of the cylinder from the experimental Case 7 and Case 8.
Transform algorithm and the discriminating statistic is chosen to be the correlation dimension of the time series data. The Fourier Transform algorithm and correlation dimension were successfully applied in the surrogate data testing for the chaotic signals of some mechanical systems with nonlinear elements in [15] and [28].

For each experimental case, the correlation dimensions are calculated for the original data and for 50 sets of the corresponding surrogate data. The calculations of the correlation dimensions for all cases are implemented in the time delay phase spaces reconstructed by the same embedding dimension. The method for calculating the correlation dimension based on the correlation sum is discussed intensively in [14,25] and [26].

Fig. 14 shows the correlation dimensions of all 9 cases together with the mean values of the correlation dimension from the corresponding surrogate data with 3σ error margins. Evidently, the responses in Case 5, 6, 8 and 9 are concluded as results of nonlinear processes, i.e. chaotic systems. The responses in the remaining cases support the null hypothesis; therefore, those are attributed to linearly noise stochastic processes.

4.3. Calculation of the largest Lyapunov exponent

A similar procedure as presented in Section 3 is applied to evaluate the largest Lyapunov exponents of the system for the cases (5, 6, 8 and 9) that exhibit the chaotic responses. Fig. 15 shows the AMI results to estimate the time delay. It can be observed that the AMI suggests a uniform first local minimum at around 0.4 s, which is in the same order of the time delay estimated for the simulated data.

Subsequently, Fig. 16 depicts the value $E_1(d)$ utilized to determine the embedding dimensions to reconstruct the time delay phase spaces for the experimental data. It can be seen that the qualified embedding dimensions for these cases are 5 or 6, while an embedding dimension of 4 is sufficient for the simulated data. This can be understood as for noise-contaminated data, a higher embedding dimension is required to completely unfold the attractor.

Based on the estimated time delay and embedding dimensions, the phase spaces are constructed and the largest Lyapunov exponents are calculated. Table 2 lists the estimated largest Lyapunov exponents of the chaotic cases (the numbers between brackets indicate the case numbers listed in Table 1). In line with the observation results from simulation, the largest Lyapunov exponent decreases when the damping imposed to the system increases. The same thing is observed when the bistable gap decreases.

In order to validate results further, two additional experimental cases were utilized. The first one (Case 10) aims to re-validate the
largest Lyapunov exponent calculation at low water velocity, with parameters of interest $\zeta=\zeta_0$, $\alpha=0.2$ and $U=0.12$ m/s. The purpose of the second set (Case 11) is to validate the chaotic existence on different normalized bistable gap with the following parameters $\zeta=\zeta_0$, $\alpha=0.4$ and $U=0.18$ m/s (see Table 3).

Observing the sequence of cases 10, 4, 5 and 6, we can conclude that when the values of damping and bistable gap are fixed, the system response is regular at low water flow velocities (Case 10 and 4). When water flow velocity is increased, chaotic response is built up in the system (Case 5). However, it also implies that the chaotic degree is reduced when the water flow velocity is further increased (Case 6). Similarly, looking at the series of cases 2, 5 and 11, where the values of damping and water flow velocity are fixed, the increase in the bistable gap will result in the chaotic response (from Case 2-5), but when the bistable gap is extended, the chaotic degree will reduce as indicated in Case 11.

The commonality on the tendency observed from simulation results and experimental data confirms the existence of chaotic response in a bistable VIV system for various governing parameters. This information can be later used as basis knowledge in designing a VIV energy converter with a bistable stiffness element with optimum harvested power while mitigating the negative effect of chaotic vibrations.

5. Conclusions

In this paper, chaotic vibrations in a VIV energy converter system enhanced by a bistable stiffness element were quantified in a wide range of the governing parameters including damping and bistable gap. The quantification was based on the quantity Lyapunov exponent, which is a reliable tool to measure the chaotic degree in nonlinear dynamic systems. In the first place, the VIV system was simulated using the wake oscillator model and the chaotic degree of the responses were quantified after the reconstruction of a time delay phase space. Subsequently, some conclusions drawn from the simulation results were verified on experimental data that was obtained using a dedicated computer-based force-feedback testing platform.

However, in experimental cases, the quantification of chaotic responses cannot be carried out in a straightforward manner, due to the presence of noise from the measurement. There is no clear distinction on the responses, whether the irregularity is attributed from the linear stochastic process (noise) or, indeed, from nonlinear processes. The chaotic response as a result of nonlinear dynamics is confirmed using the surrogate data testing with correlation dimension as the discriminating statistic.

Based from the observation on the experimental data, it is confirmed that there exists some ranges of governing parameters for which chaotic responses could occur. This finding agrees with that observed in the simulation results. Bistable stiffness on the VIV energy converter, on one side, is considered to be a potential approach to enhance the harvested power. However, the presence of chaotic responses will limit the performance of the system. Therefore, the bifurcation information, which essentially shows the map of when chaotic response will occur, is seen to be a potential tool as a primary reference in designing a bistable VIV energy converter.

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Table 2
Estimated largest Lyapunov exponents, $\lambda$ (bit/s), of the chaotic cases.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$U=0.13$ m/s</th>
<th>$U=0.18$ m/s</th>
<th>$U=0.20$ m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta=\zeta_0$</td>
<td>$\alpha=0.1$</td>
<td>(1)</td>
<td>--</td>
</tr>
<tr>
<td>$\zeta=\zeta_0$</td>
<td>$\alpha=0.2$</td>
<td>(4)</td>
<td>--</td>
</tr>
<tr>
<td>$\zeta=20\zeta_0$</td>
<td>$\alpha=0.1$</td>
<td>(7)</td>
<td>--</td>
</tr>
</tbody>
</table>

Fig. 16. The value $E_1(d)$ of the time series displacement of the cylinder from the chaotic cases.
Table 3
Estimated largest Lyapunov exponents, $\lambda$ (bit/s), of the additional experimental cases.

<table>
<thead>
<tr>
<th>Case no.</th>
<th>Parameters</th>
<th>$\lambda$ (bit/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\xi_{\xi_{0}}$</td>
<td>$\alpha=0.2$</td>
</tr>
<tr>
<td>11</td>
<td>$\xi_{\xi_{0}}$</td>
<td>$\alpha=0.4$</td>
</tr>
</tbody>
</table>

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References


