Notes on Cauchy principal and Hadamard finite-part integrals
by WT Ang
11 May 2009

Cauchy principal integrals of the form

$$\mathcal{C}\int_a^b \frac{p(t)dt}{t-x} \text{ (for } a < x < b). \quad (1)$$

appear in the formulations of many problems in physical and engineering sciences. We use the symbol $\mathcal{C}$ to indicate that the integral is improper and must be defined in a special way, since the integrand is ill defined at $t = x$, where x is a point within the interval of integration.

The integral in (1) can be defined in a limiting sense in two different ways as follows:

$$\mathcal{C}\int_a^b \frac{p(t)dt}{t-x} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}\left(\int_a^{x-\epsilon} \frac{p(t)dt}{t-x} + \int_{x+\epsilon}^b \frac{p(t)dt}{t-x}\right), \quad (2)$$

$$\mathcal{C}\int_a^b \frac{p(t)dt}{t-x} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)p(t)dt}{(t-x)^2+\epsilon^2}. \quad (3)$$

For our discussion here, we assume that $p(t)$ can be expanded as Taylor series about $t = x$ (for $a < x < b$), that is,

$$p(t) = p(x) + \sum_{m=1}^{\infty} \frac{p^{(m)}(x)}{m!}(t-x)^m. \quad (4)$$

With (4), we can show the right hand sides of (2) and (3) are given by the same series, that is, we can show that the two definitions of the Cauchy principal integrals are equivalent.

A question that arises is, "What do we get if we differentiate the Cauchy principal integral in (1) with respect to x?"

Using the definition in (2) and the series (4) for $p(t)$, we can differentiate the improper integral in (1) with respect to x as follows:

$$\begin{aligned}\frac{d}{dx}[\mathcal{C}\int_a^b \frac{p(t)dt}{t-x}] &= \frac{d}{dx}\lim_{\epsilon\to 0^+}(\int_a^{x-\epsilon}\frac{p(t)dt}{t-x}+\int_{x+\epsilon}^b\frac{p(t)dt}{t-x}) \\ &= \frac{d}{dx}(p(x)[\ln(b-x)-\ln(-a+x)]) \\ &\quad +\frac{d}{dx}\sum_{m=1}^{\infty}\frac{p^{(m)}(x)}{m!m}[(b-x)^m-(a-x)^m]). \qquad (5)\end{aligned}$$

What does the expression on the last two lines of (5) represent? To find out, let us consider the divergent integral given by

$$\int_a^b \frac{p(t)dt}{(t-x)^2} \text{ (for } a<x<b). \qquad (6)$$

If we attempt to express the integral in (6) in terms of a limit by excluding an infinitesimal part containing $t=x$ from the interval of integration and by using (4), we find that

$$\begin{aligned}\int_a^b \frac{p(t)dt}{(t-x)^2} &= \lim_{\epsilon\to 0^+}(\int_a^{x-\epsilon}\frac{p(t)dt}{(t-x)^2}+\int_{x+\epsilon}^b\frac{p(t)dt}{(t-x)^2}) \\ &= \lim_{\epsilon\to 0^+}\frac{2p(x)}{\epsilon}+F(x), \qquad (7)\end{aligned}$$

where $F(x)$ is given by

$$\begin{aligned}F(x) &= p(x)(\frac{1}{a-x}-\frac{1}{b-x}) \\ &\quad +p'(x)[\ln(b-x)-\ln(-a+x)] \\ &\quad +\sum_{m=1}^{\infty}\frac{p^{(m+1)}(x)}{(m+1)!m}[(b-x)^m-(a-x)^m]). \qquad (8)\end{aligned}$$

The function $F(x)$ is called the *finite-part* of the divergent integral in (6) (that is, the part that does not 'blow up' as ϵ approaches 0) and is denoted by

$$\mathcal{H}\int_a^b \frac{p(t)dt}{(t-x)^2}, \tag{9}$$

that is, we define

$$\mathcal{H}\int_a^b \frac{p(t)dt}{(t-x)^2} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}(\int_a^{x-\epsilon} \frac{p(t)dt}{(t-x)^2} + \int_{x+\epsilon}^b \frac{p(t)dt}{(t-x)^2} - \frac{2p(x)}{\epsilon}) \text{ for } a < x < b. \tag{10}$$

In the literature, the quantity in (9), as defined formally by (10), is called a Hadamard finite-part integral. Because of its integrand, the term "hypersingular integral" is also used to describe such an integral.

Going back to (5), we can show that

$$\begin{aligned}
& \frac{d}{dx}(p(x)[\ln(b-x) - \ln(-a+x)]) \\
& +\frac{d}{dx}(\sum_{m=1}^{\infty} \frac{p^{(m)}(x)}{m!m}[(b-x)^m - (a-x)^m]) \\
= \; & p(x)[\frac{1}{x-b} - \frac{1}{x-a}] + p'(x)[\ln(b-x) - \ln(-a+x)]) \\
& +\sum_{m=1}^{\infty} \frac{1}{m!m}\{-p^{(m)}(x)m[(b-x)^{m-1} - (a-x)^{m-1}] \\
& +p^{(m+1)}(x)[(b-x)^m - (a-x)^m]\} \\
& \vdots \text{ (after some re-arrangement of the terms)} \\
= \; & F(x).
\end{aligned}$$

Thus, we conclude that

$$\frac{d}{dx}[\mathcal{C}\int_a^b \frac{p(t)dt}{t-x}] = \mathcal{H}\int_a^b \frac{p(t)dt}{(t-x)^2} \text{ for } a < x < b. \tag{11}$$

An alternative definition for the Hadamard finite-part integral is given by Ang and Clements[1] as

$$\mathcal{H}\int_a^b \frac{p(t)dt}{(t-x)^2} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}\left(\int_a^b \frac{(t-x)^2 p(t)dt}{[(t-x)^2+\epsilon^2]^2} - \frac{\pi}{2\epsilon}p(x)\right) \text{ for } a < x < b. \quad (12)$$

The above alternative definition can be shown to be equivalent to (10) if we replace $p(t)$ on the right hand side of (12) by the series in (4).

How can the Hadamard finite-part integral in (9) be evaluated? It can be easily evaluated if we can find an antiderivative of its integrand. The usual fundamental theorem for evaluating Riemman integrals also work for Hadamard finite-part integrals, that is,

$$\mathcal{H}\int_a^b \frac{p(t)dt}{(t-x)^2} = G(b) - G(a) \text{ if } G'(t) = \frac{p(t)}{(t-x)^2}.$$

(Can you see why?) For example,

$$\mathcal{H}\int_0^2 \frac{dt}{(t-1)^2} = \left.\frac{1}{1-t}\right|_{t=0}^{t=2} = -1 - 1 = -2.$$

Note that the usual numerical integration rules (such as the trapezoidal rule) for Riemann integrals however do not work for Hadamard finite-part integrals (as Hadamard finite-part integrals do not give the "area underneath a curve").

Hypersingular integral formulation of a simple crack problem

We will show how the Hadamard finite-part integral defined in (10) or (12) appears in the formulation of crack problems. For clarity, we consider a simple mode III crack problem which requires us to solve the two-dimensional Laplace's equation for $\phi(x, y)$ on the whole of the Oxy plane containing a finite cut (a crack) in the region $-a < x < a$, $y = 0$. The solution $\phi(x, y)$ is

[1] Ang WT, Clements DL. Hypersingular integral equations for a thermoelastic problem of multiple planar cracks in an anisotropic medium. *Engineering Analysis with Boundary Elements* **23** (1999) 713-720.

required to satisfy the following conditions:

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = q(x) \text{ for } -a < x < a, \tag{13}$$

$$\phi \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty, \tag{14}$$

where $q(x)$ is a suitably give function.

To solve the problem, we take the boundary integral equation[2] for the two-dimensional Laplace's equation as our starting point, that is,

$$\phi(\xi,\eta) = \int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)$$
$$\text{for } (\xi,\eta) \text{ lying in the interior of the solution domain,} \tag{15}$$

where

$$\Phi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln[(x-\xi)^2 + (y-\eta)^2],$$
$$\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) = \frac{(x-\xi)n_x(x,y) + (y-\eta)n_y(x,y)}{2\pi[(x-\xi)^2 + (y-\eta)^2]}, \tag{16}$$

where $[n_x(x,y), n_y(x,y)]$ is the outward unit normal vector to C at the point (x,y).

Here the boundary C comprises two parts: C_∞ (the boundary at infinity) and L (the crack). The crack L has two opposite faces: L^+ (upper face, $-a < x <, y = 0^+$) and L^- (lower face, $-a < x <, y = 0^-$). The boundary integral equation in (15) can be rewritten as

$$\phi(\xi,\eta) = \int_{C_\infty} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)$$
$$+\int_{-a}^{a} [-\phi(x,0^+)\left.\frac{\partial}{\partial y}(\Phi(x,y;\xi,\eta))\right|_{y=0^+}$$
$$+\phi(x,0^-)\left.\frac{\partial}{\partial y}(\Phi(x,y;\xi,\eta))\right|_{y=0^-}]dx$$
$$+\int_{-a}^{a} [\Phi(x,0^+;\xi,\eta) - \Phi(x,0^-;\xi,\eta)]q(x)dx \tag{17}$$

[2] W. T. Ang, *A Beginner's Course in Boundary Element Methods*, Universal Publishers, 2007.

for (ξ, η) lying in the interior of the Oxy plane with a finite cut $-a < x < a$, $y = 0$.

In view of the far field condition in (14), we assume that ϕ behaves as $O([x^2 + y^2]^{-\alpha})$ $(\alpha > 0)$ for large $x^2 + y^2$. It can then be shown that

$$\int\limits_{C_\infty} [\phi(x, y)\frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta)) - \Phi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, y))]ds(x, y) = 0.$$

For the crack problem, ϕ may jump across the opposite crack faces, that is, $\phi(x, 0^+)$ and $\phi(x, 0^-)$ are not necessarily equal to each other. From (16), we can see that

$$\begin{aligned} \Phi(x, 0^+; \xi, \eta) &= \Phi(x, 0^-; \xi, \eta), \\ \frac{\partial}{\partial y}(\Phi(x, y; \xi, \eta))\bigg|_{y=0^+} &= \frac{\partial}{\partial y}(\Phi(x, y; \xi, \eta))\bigg|_{y=0^-}. \end{aligned}$$

Thus, for (ξ, η) in the Oxy plane with a cut $-a < x < a$, $y = 0$, (17) reduces to

$$\phi(\xi, \eta) = \eta \int\limits_{-a}^{a} \frac{\Delta\phi(x)dx}{2\pi[(x - \xi)^2 + \eta^2]} \tag{18}$$

where

$$\Delta\phi(x) = \phi(x, 0^+) - \phi(x, 0^-) \text{ for } -a < x < a. \tag{19}$$

Note $\Delta\phi(x)$ is an unknown function to be determined.

If we expand $\Delta\phi(x)$ in terms of its Taylor series about $x = \xi$, from (18), we can show that, for $-a < \xi < a$,

$$\begin{aligned} \phi(\xi, 0^+) &= \lim_{\eta \to 0^+} \eta \int\limits_{-a}^{a} \frac{\Delta\phi(x)dx}{2\pi[(x - \xi)^2 + \eta^2]} = \frac{1}{2}\Delta\phi(\xi), \\ \phi(\xi, 0^-) &= \lim_{\eta \to 0^+} \eta \int\limits_{-a}^{a} \frac{\Delta\phi(x)dx}{2\pi[(x - \xi)^2 + \eta^2]} = -\frac{1}{2}\Delta\phi(\xi) \end{aligned}$$

Note that $\phi(\xi, 0^+) = \phi(\xi, 0^-) = 0$ for $\xi \in (-\infty, -a) \cup (a, \infty)$ (on the x axis outside the crack).

If we differentiate (18) partially with respect to η, we obtain

$$\frac{\partial}{\partial\eta}[\phi(\xi,\eta)] = \int_{-a}^{a} \frac{(x-\xi)^2\Delta\phi(x)dx}{2\pi[(x-\xi)^2+\eta^2]^2} - \eta^2\int_{-a}^{a}\frac{\Delta\phi(x)dx}{2\pi[(x-\xi)^2+\eta^2]^2}. \tag{20}$$

By expanding $\Delta\phi(x)$ in terms of its Taylor series about $x = \xi$, from (20) together with the definition (12), it can be shown that

$$\begin{aligned}
\left.\frac{\partial}{\partial\eta}[\phi(\xi,\eta)]\right|_{\eta=0^+} &= \lim_{\eta\to 0^+}\left(\int_{-a}^{a}\frac{(x-\xi)^2\Delta\phi(x)dx}{2\pi[(x-\xi)^2+\eta^2]^2} - \frac{\Delta\phi(\xi)}{4\eta}\right) \\
&= \frac{1}{2\pi}\lim_{\eta\to 0^+}\left(\int_{-a}^{a}\frac{(x-\xi)^2\Delta\phi(x)dx}{[(x-\xi)^2+\eta^2]^2} - \frac{\pi\Delta\phi(\xi)}{2\eta}\right) \\
&= \frac{1}{2\pi}\mathcal{H}\int_{-a}^{a}\frac{\Delta\phi(x)dx}{(x-\xi)^2} \text{ for } -a<\xi<a.
\end{aligned}$$

It follows that the condition in (13) gives rise

$$\frac{1}{2\pi}\mathcal{H}\int_{-a}^{a}\frac{\Delta\phi(x)dx}{(x-\xi)^2} = q(\xi) \text{ for } -a<\xi<a, \tag{21}$$

a Hadamard finite-part (hypersingular) integral equation with $\Delta\phi(x)$ as unknown function to be determined. For crack problems, the unknown function $\Delta\phi(x)$ takes the form $\Delta\phi(x) = \sqrt{a^2-x^2}\psi(x)$ for $-a < x < a$, and for given $q(x)$ it may be possible to invert (21) to obtain $\psi(x)$ analytically. Even if we do not know how to invert (21) analytically, there are numerical methods[3] for determining $\psi(x)$ from the hypersingular integral equation.

[3] See, for example, Kaya A, Erdogan F. On the solution of integral equations with strongly singular kernels. *Quarterly of Applied Mathematics* **45** (1987) 105-122.