Boundary Integral Equation for Axisymmetric Potential Problem by W. T. Ang 26 February 2009

The boundary integral equation for the three-dimensional potential problem is given in Chapter 6 of the book "*A Beginner's Course in Boundary Element Methods*" as

$$\lambda(\xi,\eta,\zeta)\phi(\xi,\eta,\zeta) = \iint_S (\phi(x,y,z)\frac{\partial}{\partial n}[\Phi_{\rm 3D}(x,y,z;\xi,\eta,\zeta)] - \Phi_{\rm 3D}(x,y,z;\xi,\eta,\zeta)\frac{\partial}{\partial n}[\phi(x,y,z)])ds(x,y,z), \quad (1)$$

where ϕ satisfies the three-dimensional Laplace's equation in the region R bounded by a closed surface S, $\lambda(\xi,\eta,\zeta)$ is defined by

$$\lambda(\xi,\eta,\zeta) = \begin{cases} 0 & \text{if } (\xi,\eta,\zeta) \notin R\cup S, \\ 1/2 & \text{if } (\xi,\eta,\zeta) \text{ lies on a smooth part of } S, \\ 1 & \text{if } (\xi,\eta,\zeta) \in R, \end{cases} \quad (2)$$

and $\Phi_{\rm 3D}$ is the fundamental solution given by

$$\Phi_{\rm 3D}(x,y,z;\xi,\eta,\zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}. \quad (3)$$

Let us now consider the axisymmetric case in which the surface S of the solution domain can be generated by rotating a curve Γ about the z-axis by an angle of 360°. For example, if S is the sphere $x^2+y^2+(z-2)^2=1$ (sphere of center $(0,0,2)$ and radius 1) then we can generate the surface S by rotating the semi-circle $x^2+(z-2)^2=1$, $x\geq 0$, on the Oxz plane by an angle of 360° about the z-axis. For such a surface S and for ϕ which does not change with the polar coordinate θ but only with r and z, that is, for an axisymmetric problem, the boundary integral over S in (1) can be reduced to an integral over the curve Γ as explained below. (In cylindrical polar coordinates, points can be described using (r,θ,z) instead of (x,y,z), where $x=r\cos\theta$ and $y=r\sin\theta$.)

Firstly, let us define

$$\begin{aligned} \phi^*(r,\theta,z) &= \phi(r\cos\theta, y\sin\theta, z) \\ p^*(r,\theta,z) &= \frac{\partial}{\partial n}[\phi(x,y,z)]\bigg|_{(x,y,z)=(r\cos\theta,y\sin\theta,z)}. \end{aligned}$$

For an axisymmetric problem, ϕ^* is independent of θ and we can write $\phi^*(r,z)$. We will show now that, for axisymmetric problem, p^* also depends only on r and z. We have:

$$\begin{aligned} \frac{\partial}{\partial n}[\phi(x,y,z)] &= n_x\frac{\partial\phi}{\partial x} + n_y\frac{\partial\phi}{\partial y} + n_x\frac{\partial\phi}{\partial z} \\ &= n_x(\frac{\partial\phi^*}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi^*}{\partial\theta}\frac{\partial\theta}{\partial x}) \\ &\quad + n_y(\frac{\partial\phi^*}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi^*}{\partial\theta}\frac{\partial\theta}{\partial y}) + n_z\frac{\partial\phi^*}{\partial z}. \end{aligned}$$

If we introduce a local (polar) coordinate system with base vectors $\underline{\mathbf{e}}_r$, $\underline{\mathbf{e}}_\theta$ and $\underline{\mathbf{e}}_z = \underline{\mathbf{k}}$, then the unit normal vector is given by $n_r\underline{\mathbf{e}}_r + n_\theta\underline{\mathbf{e}}_\theta + n_z\underline{\mathbf{e}}_z$. On a fixed plane $z = c$ (constant), if the body is axisymmetric, the components n_r, n_θ and n_z do not change with θ, but n_x and n_y change with θ. It may be shown that

$$\begin{aligned} n_x &= n_r \cos\theta - n_\theta \sin\theta \\ n_y &= n_r \sin\theta + n_\theta \cos\theta. \end{aligned}$$

It follows that (for the axisymmetric problem)

$$\begin{aligned} p^* &= (n_r\cos\theta - n_\theta\sin\theta)[\cos\theta\frac{\partial\phi^*}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\phi^*}{\partial\theta}] \\ &\quad +(n_r\sin\theta + n_\theta\cos\theta)[\sin\theta\frac{\partial\phi^*}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\phi^*}{\partial\theta}] + n_z\frac{\partial\phi^*}{\partial z} \\ &= n_r\frac{\partial\phi^*}{\partial r} + \frac{1}{r}n_\theta\frac{\partial\phi^*}{\partial\theta} + n_z\frac{\partial\phi^*}{\partial z}. \end{aligned}$$

Since ϕ^*, n_r and n_z are independent of θ, we find that $p^* = n_r\partial\phi^*/\partial r + n_z\partial\phi^*/\partial z$ is also independent of θ.

For convenience, we will now drop the asterik $*$ and write $\phi^*(r,z)$ as merely $\phi(r,z)$ and $p^*(r,z)$ as $p(r,z)$.

For S which is symmetrical about the z-axis, the infinitesimal area $ds(x,y,z)$ in (1) can be written as

$$ds(x,y,z) = r\,d\ell\,d\theta. \tag{4}$$

where $d\ell$ is the length of an infinitesimal portion of the curve Γ.

Consider now (1) for the axisymmetric potential problem. For a point $(\xi,\eta,\zeta) = (r_0\cos\theta_0, r_0\sin\theta_0, z_0)$ on the Oxz plane (where $y = 0$ or $\theta_0 = 0$), we can rewrite (1) as

$$\begin{aligned} \lambda(r_0,z_0)\phi(r_0,z_0) &= \iint\limits_S (\phi(r,z)\frac{\partial}{\partial n}[\Phi_{\mathrm{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\ &\quad -\Phi_{\mathrm{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)p(r,z))r\,d\ell\,d\theta, \end{aligned} \tag{5}$$

where $\lambda(r_0,z_0) = 1/2$ if (r_0,z_0) lies on a smooth part of Γ and $\lambda(r_0,z_0) = 1$ if (r_0,z_0) lies in the interior of the solution domain on the Oxz plane.

We need to integrate with respect to θ from 0 to 2π as the complete surface S is obtained by rotating Γ by an angle of 360°. Note that θ appears only in the function Φ_{3D} and not in $\phi(r,z)$. The integration with respect to r and z (that is, with respect to ℓ) is over the curve Γ. Thus, we can rewrite (5) as

$$\begin{aligned} \lambda(r_0,z_0)\phi(r_0,z_0) &= \int\limits_\Gamma (\phi(r,z)\Psi_{\mathrm{axis}}(r,z;r_0,z_0;n_r,n_z) \\ &\quad -\Phi_{\mathrm{axis}}(r,z;r_0,z_0)p(r,z))r d\ell(r,z), \end{aligned} \tag{6}$$

where

$$\begin{aligned}
\Phi_{\text{axis}}(r,z;r_0,z_0) &= \int_0^{2\pi} \Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0) d\theta \\
&= -\frac{1}{4\pi}\int_0^{2\pi} \frac{1}{\sqrt{(r\cos\theta - r_0)^2 + r^2\sin^2\theta + (z-z_0)^2}} d\theta
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4\pi}\int_0^{2\pi} \frac{1}{\sqrt{r^2 + r_0^2 + (z-z_0)^2 - 2rr_0\cos\theta}} d\theta \\
&= -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \int_0^{2\pi} \frac{1}{4\sqrt{1 - \frac{2rr_0(1+\cos\theta)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}} d\theta
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \int_0^{\pi} \frac{1}{2\sqrt{1 - \frac{2rr_0(1+\cos(2t))}{r^2+r_0^2+(z-z_0)^2+2rr_0}}} dt \\
&= -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \int_0^{\pi} \frac{1}{2\sqrt{1 - \frac{4rr_0\cos^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}} dt
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{4rr_0\cos^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}} dt \\
&= -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{4rr_0\sin^2(t)}{r^2+r_0^2+(z-z_0)^2+2rr_0}}} dt.
\end{aligned}$$

If we define the function $K(m)$ as

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m\sin^2(t)}} dt, \tag{7}$$

then we can write

$$\Phi_{\text{axis}}(r,z;r_0,z_0) = -\frac{1}{\pi\sqrt{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}} K\left(\frac{4rr_0}{r^2 + r_0^2 + (z-z_0)^2 + 2rr_0}\right). \tag{8}$$

In mathematics, K is a special function and is called the *complete elliptic integral of the first kind*. There is a simple approximate but accurate formula in

Abramowitz and Stegun's *Handbook of Mathematical Functions* for evaluating $K(m)$. Some mathematical softwares may have inbuilt functions for calculating $K(m)$. Note that $0 \leq \frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0} \leq 1$ and $K(m)$ is undefined for $m = 1$.

Also:

$$\begin{aligned}\Psi_{\text{axis}}(r,z;r_0,z_0;n_r,n_z) &= \int_0^{2\pi} \frac{\partial}{\partial n}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)]d\theta \\ &= \int_0^{2\pi} (n_r \frac{\partial}{\partial r}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\ &\quad +\frac{1}{r}n_\theta \frac{\partial}{\partial \theta}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)] \\ &\quad +n_z \frac{\partial}{\partial z}[\Phi_{\text{3D}}(r\cos\theta, r\sin\theta, z; r_0, 0, z_0)])d\theta \qquad (9) \\ &= n_r \frac{\partial}{\partial r}[\Phi_{\text{axis}}(r,z;r_0,z_0)] + n_z \frac{\partial}{\partial z}[\Phi_{\text{axis}}(r,z;r_0,z_0)]\end{aligned}$$

For the axisymmetric body, can you see why $n_\theta = 0$?

There is this relationship:

$$\frac{d}{dm}(K(m)) = \frac{1}{2m}(\frac{E(m)}{1-m} - K(m)), \tag{10}$$

where

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2 t}dt. \tag{11}$$

Note that $E(m)$ is known as the *complete elliptic integral of the second kind*. Details on computing $E(m)$ are available in Abramowitz and Stegun.

From (9) and (10), it can be shown that

$$\begin{aligned}&\Psi_{\text{axis}}(r,z;r_0,z_0;n_r,n_z) \\ = &-\frac{1}{\pi\sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0}} \\ &\times\{\frac{n_r}{2r}[\frac{r_0^2-r^2+(z-z_0)^2}{r^2+r_0^2+(z-z_0)^2-2rr_0}E(\frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0}) \\ &-K(\frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0})] \\ &+n_z\frac{z_0-z}{r^2+r_0^2+(z-z_0)^2-2rr_0}E(\frac{4rr_0}{r^2+r_0^2+(z-z_0)^2+2rr_0})\}. \qquad (12)\end{aligned}$$

If we use (6) to devise a boundary element method for solving the axisymmetric potential problem, we only discretize the curve Γ on the rz space (that is, the Oxz plane) instead of the three-dimensonal surface S.