Generalized Balanced Tournament Packings and Optimal Equitable Symbol Weight Codes for Power Line Communications

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Abstract: Generalized balanced tournament packings (GBTPs) extend the concept of generalized balanced tournament designs introduced by Lamken and Vanstone (1989). In this paper, we establish the connection between GBTPs and a class of codes called equitable symbol weight codes (ESWCs). The latter were recently demonstrated to optimize the performance against narrowband noise in a general coded modulation scheme for power line communications. By constructing classes of GBTPs, we establish infinite families of optimal ESWCs with code lengths greater than alphabet size and whose narrowband noise error-correcting capability to code length ratios do not diminish to zero as the length grows. © 2014 Wiley Periodicals, Inc. J. Combin. Designs 00: 1–32, 2014

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1. INTRODUCTION

Power line communications (PLCs) is a technology that enables the transmission of data over electric power lines. It was started in the 1910s for voice communication [28], and used in the 1950s in the form of ripple control for load and tariff management in power distribution. With the emergence of the Internet in the 1990s, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides “last mile” connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [15, 17, 25, 34]). However, power lines present a difficult communications environment and overcoming permanent narrowband disturbance has remained a challenging problem [2, 26, 30]. Vinck [30] addressed this problem through the use of a coded modulation scheme based on permutation codes. More recently, Chee et al. [8] extended Vinck’s analysis to general block codes and motivated the study of equitable symbol weight codes (ESWCs).

Relatively little is known about optimal ESWCs, other than those that correspond to permutation codes, injection codes, and frequency permutation arrays. In particular, only six infinite families of optimal ESWCs with code length greater than alphabet size are known. These have all been constructed by Ding and Yin [13], and Huczynska and Mullen [18] as frequency permutation arrays and they meet the Plotkin bound. One drawback with the code parameters of these families is that the narrowband noise error-correcting capability to length ratio diminishes as length grows.

In this paper, we construct infinite families of optimal ESWCs whose code lengths are larger than alphabet size and whose narrowband noise error-correcting capability to length ratios tend to a positive constant as code length grows. These families of codes all attain the generalized Plotkin bound. Our results are based on the construction of equivalent combinatorial designs called generalized balanced tournament packings (GBTPs).

GBTPs extend the concept of generalized balanced tournament designs (GBTDs) introduced by Lamken and Vanstone [19]. GBTDs have been extensively studied [9, 11, 20–22, 33] and are useful in the constructions of resolvable, near-resolvable, doubly resolvable, and doubly near-resolvable balanced incomplete block designs [20, 23, 24]. Using the classical correspondence given by Semakov and Zinoviev [29] (see also [4, 12, 33]), we construct optimal families of ESWCs from certain families of GBTPs. We establish existence results for these families of GBTPs by borrowing standard recursion and direct construction methods from combinatorial design theory.

The paper is organized as follows. In Section 2, we introduce ESWCs and survey the known results on optimal codes. In Section 3, we introduce GBTPs and establish the equivalence between GBTPs and ESWCs. At the end of the section, we establish two classes of GBTPs that correspond to optimal ESWCs. In Sections 4–7, we settle the existence of these two classes of GBTPs. Section 4 outlines the general strategy, while Sections 5 and 6 provide recursive and direct constructions, respectively.

Some of the results of the paper have been initially reported at IEEE International Symposium on Information Theory 2012 [7], and the present paper contains detailed proofs and includes a new existence result on a family of GBTPs with block size 2 and 3.
2. PRELIMINARIES

2.1. Notation

For positive integer \( m \) and prime power \( q \), denote the ring \( \mathbb{Z}/m\mathbb{Z} \) by \( \mathbb{Z}_m \) and the finite field of \( q \) elements by \( \mathbb{F}_q \). Let \( \mathbb{Z}_{>0} \) denote the set of positive integers. Let \([m]\) denote the set \( \{1, 2, \ldots, m\} \). We use angled brackets (⟨ and ⟩) for multisets. Disjoint set union is depicted using \( \sqcup \). For sets \( A \) and \( B \), an element \((a, b) \in A \times B\) is sometimes written as \( ab \) for succinctness.

A set system is a pair \( S = (X, A) \), where \( X \) is a finite set of points and \( A \subseteq \mathcal{P}(X) \). Elements of \( A \) are called blocks. The order of \( S \) is the number of points in \( X \), and the size of \( S \) is the number of blocks in \( A \). Let \( K \) be a set of nonnegative integers. The set system \((X, A)\) is \( K \)-uniform if \( |A| \in K \) for all \( A \in A \).

2.2. Equitable Symbol Weight Codes

Let \( \Sigma \) be a set of \( q \) symbols. A \( q \)-ary code of length \( n \) over the alphabet \( \Sigma \) is a subset \( C \subseteq \Sigma^n \). Elements of \( C \) are called codewords. The size of \( C \) is the number of codewords in \( C \). For \( i \in [n] \), the \( i \)th coordinate of a codeword \( u \in C \) is denoted \( u_i \), so that \( u = (u_1, u_2, \ldots, u_n) \). Denote the frequency of symbol \( \sigma \in \Sigma \) in codeword \( u \in \Sigma^n \) by \( w_\sigma(u) \), that is, \( w_\sigma(u) = |\{i \in [n] : u_i = \sigma\}| \).

An element \( u \in \Sigma^n \) is said to have equitable symbol weight if \( w_\sigma(u) \in \lfloor n/q \rfloor, \lceil n/q \rceil \) for any \( \sigma \in \Sigma \). If all the codewords of \( C \) have equitable symbol weight, then the code \( C \) is called an ESWC. Consider the usual Hamming distance defined on codewords and codes and let \( d \) denote the minimum distance of a code \( C \). In addition, consider the following parameter.

**Definition 2.1.** Let \( C \) be a \( q \)-ary code with minimum distance \( d \). The narrowband noise error-correcting capability of \( C \) is

\[
c(C) = \min\{e : E_C(e) \geq d\},
\]

where \( E_C \) is a function \( E_C : [q] \to [n] \), given by

\[
E_C(e) = \max_{\Gamma \subseteq \Sigma} \max_{c \in C} \left\{ \sum_{\sigma \in \Gamma} w_\sigma(c) \right\}.
\]

Chee et al. [8] established that a code \( C \) can correct up to \( c(C) - 1 \) narrowband noise errors and demonstrated that an ESWC maximizes the quantity \( c(C) \), for fixed \( n, d, \) and \( q \).

Henceforth, only ESWCs are considered. A \( q \)-ary ESWC of length \( n \) having minimum distance \( d \) is denoted \( \text{ESWC}(n, d)_q \). Denote the maximum size of an ESWC \( \text{ESWC}(n, d)_q \) by \( \Lambda_q^{\text{ESW}}(n, d) \). Any ESWC \( \text{ESWC}(n, d)_q \) of size \( \Lambda_q^{\text{ESW}}(n, d) \) is said to be optimal. Taken as a \( q \)-ary code of length \( n \), an optimal ESWC \( \text{ESWC}(n, d)_q \) satisfies the generalized Plotkin bound [3].
Theorem 2.2 (Generalized Plotkin bound). If there is an ESWC \((n, d)_q\) of size \(M\), then
\[
\binom{M}{2}^{d} \leq n \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M_i M_j,
\]
where \(M_i = \lfloor (M + i)/q \rfloor\). If \(q\) divides \(M\) and \((\binom{M}{2})^d = n \binom{q}{2}(M/q)^2\), then \(C\) is optimal.

In the rest of this paper, ESWCs whose sizes attain the generalized Plotkin bound are constructed. In particular, the following is established.

Theorem 2.3. The following holds.

(i) \(A^{ESW}_q(2q - 1, 2q - 2) = \begin{cases} 3, & q = 2, \\ 2q, & q \geq 3. \end{cases}\)

(ii) \(A^{ESW}_q(3q - 1, 3q - 3) = \begin{cases} 4, & q = 2, \\ 3q, & q \geq 3. \end{cases}\)

(iii) \(A^{ESW}_q(4q - 1, 4q - 4) = \begin{cases} 4q - 1, & q = 2, 3, \\ 4q, & q \geq 4. \end{cases}\)

(iv) If \(q \geq 62\) or \(q \in \{5 − 18, 30, 42, 46, 48 − 50, 54 − 57\}, A^{ESW}_q(5q - 1, 5q - 5) = 5q.\)

(v) If \(q\) is an odd prime power, \(A^{ESW}_q(q^2 - 1, q^2 - q) = q^2.\)

(vi) \(A^{ESW}_q\left(\frac{3q - 1}{2}, \frac{3q - 3}{2}\right) = \begin{cases} 4q - 6, & q = 3, 5, \\ 3q, & q \geq 7 \text{ is odd}. \end{cases}\)

(vii) \(A^{ESW}_q(2q - 3, 2q - 4) = \begin{cases} 6q - 12, & q = 3, 4, \\ 14, & q = 5, 6, \\ 2q + 1, & q \geq 7, \text{ except possibly } q \in \{12, 13\}. \end{cases}\)

Observe that any ESWC \(C\) with the above parameters must have \(c(C) = q - 1\). In Table I, we verify that \(c(C)/n\) tends to a positive constant as \(q\) grows. In the same table, we compare with known families of optimal ESWC\((n, d)_q\).
### TABLE I. Infinite families of optimal ESWC($n$, $d$)$_q$.

| ESWC($n$, $d$)$_q$ | $|C|$ | $c(C)$ | $\lim_{q \to \infty} c(C)/n$ | Remarks |
|-------------------|------|--------|-----------------|---------|
| $(n, n)_q$ for $q \geq 2$ | $q$ | $\min\|n, q\|$ | 0 | easy |
| $(3, 2)_q$ for $q \geq 3$ | $q(q - 1)$ | 2 | 0 | Injection code [14] |
| $(4, 2)_q$ for $q \geq 4, q \neq 7$ | $q(q - 1)(q - 2)$ | 2 | 0 | Injection code [14] |
| $(n, 1)_q$ for $n < q$ | $q(q - 1) \cdots (q - n + 1)$ | 1 | $1/n$ | Injection code, easy |
| $(q, 2)_q$ for $q \geq 2$ | $(qn)!/(n!)^q$ | 2 | 0 | Frequency permutation array, easy |
| $(q, 3)_q$ for $q \geq 3$ | $q!/2$ | 3 | 0 | Permutation code, easy |
| $(q, q)_q$ for prime powers $q$ | $q(q - 1)$ | $q - 1$ | 1 | Injection code [10] |
| $(n, n - 1)_q$ for $q$ sufficiently large and $n \leq q$ | $q(q - 1)$ | $n - 1$ | $1 - 1/n$ | Injection code [14] |
| $(q, q)_q$ for prime powers $q - 1$ | $q(q - 1)(q - 2)$ | $q - 2$ | 1 | Permutation code [16] |
| $(q + 1, q^2)_q$ for prime powers $q$ | $q^2$ | $q$ | 0 | Frequency permutation array [12] |
| $(q^2 - 1, 2q - 2)_q$ for even $q, q \notin \{2, 6\}$ | | | | |
| $(3q - 1, 3q - 3)_q$ for $q \geq 3$ | $2q$ | $q - 1$ | $1/2$ | Theorem 2.3 |
| $(4q - 1, 4q - 4)_q$ for $q \geq 4$ | $4q$ | $q - 1$ | $1/4$ | Theorem 2.3 |
| $(5q - 1, 5q - 5)_q$ for $q \geq 62$ | $5q$ | $q - 1$ | $1/5$ | Theorem 2.3 |
| $(q^2 - 1, q^2 - q)_q$ for $q \geq 4$ | $q^2$ | $q - 1$ | 0 | Theorem 2.3 |
| $(2q - 1, 2q - 3)_q$ for $q \geq 7$ and $q$ odd | $3q$ | $q - 1$ | $2/3$ | Theorem 2.3 |
| $(2q - 3, 2q - 4)_q$ for $q \geq 11$ | $2q + 1$ | $q - 1$ | $1/2$ | Theorem 2.3 |
In particular, only six infinite nontrivial families of optimal codes with $n > q$ are known. However, code parameters for these six families are such that their relative narrowband noise error-correcting capability to length ratios diminish to zero as $q$ grows. This is undesirable for narrowband noise correction for PLC. Hence, Theorem 2.3 provides infinite families of optimal ESWCs with code lengths larger than alphabet size and whose relative narrowband noise capability to length ratios tend to a positive constant as length grows.

These optimal ESWCs are constructed from GBTPs using the classical correspondence given by Semakov and Zinoviev [29].

We remark that GBTPs extend the concept of GBTDs and consequently Theorem 2.3 (i)–(v) follows directly from known classes of GBTDs. We explain the connection in detail in the next section.

3. CONSTRUCTIONS OF ESWCs

We first determine $A_q^{ESW}(n, d)$ for small values of $n$, $q$, and $d$. With the exception of $A_6^{ESW}(9, 8)$, an exhaustive computer search established the following values of $A_q^{ESW}(n, d)$. For $A_6^{ESW}(9, 8)$, an ESWC$(9, 8)_6$ of size 14 was found via computer search. Since an ESWC$(9, 8)_6$ of size 15 cannot exist by the generalized Plotkin bound, it follows that $A_6^{ESW}(9, 8) = 14$. We record the results of the computations in the following proposition and the corresponding optimal codes can be found at [5].

**Proposition 3.1.** The following holds:

\[
egin{align*}
A_2^{ESW}(3, 2) &= 3 & A_2^{ESW}(5, 3) &= 4 & A_2^{ESW}(7, 4) &= 7 \\
A_3^{ESW}(3, 2) &= 6 & A_5^{ESW}(4, 3) &= 6 & A_3^{ESW}(11, 8) &= 11 \\
A_4^{ESW}(5, 4) &= 12 & A_5^{ESW}(7, 6) &= 14 & A_6^{ESW}(9, 8) &= 14.
\end{align*}
\]

The rest of the paper establishes the remaining values in Theorem 2.3. To do so, we define a class of combinatorial designs that is equivalent to ESWCs.

3.1. ESWCs and GBTPs

Let $\lambda$, $v$ be positive integers and $K$ be a set of nonnegative integers. A $(v, K, \lambda)$-**packing** is a $K$-uniform set system of order $v$ such that every pair of distinct points is contained in at most $\lambda$ blocks. The value $\lambda$ is called the index of the packing. A parallel class (or resolution class) of a packing is a subset of the blocks that partitions the set of points $X$. If the set of blocks can be partitioned into parallel classes, then the packing is resolvable, and denoted by $\text{RP}(v, K, \lambda)$. An $\text{RP}(v, K, \lambda)$ is called a maximum resolvable packing, denoted by $\text{MRP}(v, K, \lambda)$, if it contains maximum possible number of parallel classes.

Furthermore, an $\text{MRP}(v, \{k\}, \lambda)$ is called a resolvable $(v, \{k\}, \lambda)$-**balanced incomplete block design**, or $\text{RBIBD}(v, k, \lambda)$ in short, if every pair of distinct points is contained in exactly $\lambda$ blocks. A simple computation gives the size of an $\text{RBIBD}(v, k, \lambda)$ to be $\frac{\lambda v(v-1)}{k(k-1)}$.

We define the combinatorial object of study in this paper. We note that this definition is a generalization of GBTDs to packings and various indices.

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1Bogdanova et al. [4] gave a survey of connection between equidistant codes and designs. Using this correspondence, Ding and Yin [12] constructed optimal constant-composition codes, while Yin et al. [33] constructed near-constant-composition codes.
Definition 3.2. Let \((X, A)\) be an \(\text{RP}(v, K, \lambda)\) with \(n\) parallel classes. Then \((X, A)\) is called a GBTP if the blocks of \(A\) are arranged into an \(m \times n\) array satisfying the following conditions:

(i) every point in \(X\) is contained in exactly one cell of each column,
(ii) every point in \(X\) is contained in either \([n/m]\) or \([n/m]\) cells of each row.

We denote such a GBTP by \(\text{GBTP}_n(K; v, m \times n)\).

Unless otherwise stated, the rows of a \(\text{GBTP}_n(K; v, m \times n)\) are indexed by \([m]\) and the columns by \([n]\).

In a \(\text{GBTP}_n(K; v, m \times n)\), given point \(x\) and column \(j\), there is a unique row that contains the point \(x\) in column \(j\). Hence, for each point \(x \in X\) of a \(\text{GBTP}_n(K; v, m \times n)\) \((X, A)\), we may correspond the codeword \(c(x) = (r_1, r_2, \ldots, r_n) \in [m]^n\), where \(r_j\) is the row in which point \(x\) appears in column \(j\). It is obvious that \(C = \{c(x): x \in X\}\) is an \(m\)-ary code of length \(n\) over the alphabet \([m]\). We note that this correspondence is precisely the one used by Semakov and Zinoviev \([29]\) to show the equivalence between equidistant codes and resolvable balanced incomplete block designs.

For distinct points \(x, y \in X\), the distance between \(c(x)\) and \(c(y)\) is the number of columns for which \(x\) and \(y\) are not both contained in the same row. Since there are at most \(\lambda\) blocks containing both \(x\) and \(y\), and that no two such blocks can occur in the same column of the \(\text{GBTP}_n(K; v, m \times n)\), the distance between \(c(x)\) and \(c(y)\) is at least \(n - \lambda\).

Next, we determine \(w_j(c(x))\), for \(x \in X\) and \(i \in [m]\). From the construction of \(c(x)\), the number of times a symbol \(i\) appears in \(c(x)\) is the number of cells in row \(i\) that contains \(x\). By the definition of a \(\text{GBTP}_n(K; v, m \times n)\), this number belongs to \([\lfloor n/m \rfloor, \lceil n/m \rceil]\). Hence, \(C\) is an ESWC of size \(v\). Finally, this construction of an ESWC from a GBTP can easily be reversed. We record these observations as:

Theorem 3.3. Let \(K\) be set of non-negative integers. Then a \(\text{GBTP}_n(K; v, m \times n)\) exists if and only if an \(\text{ESWC}(n, n - \lambda, m)\) of size \(v\) exists.

We note that the correspondence between GBTPs and ESWCs was observed by Yin et al. \([33, \text{Theorem 2.2}]\). However, in the latter paper, the class of codes constructed is called near-constant-composition codes (NCCCs). Indeed, an NCCC is a special class of ESWC and one observes that an ESWC\((n, d, q)\) is an NCCC when \(n + 1 \equiv 0\) mod \(q\).

Example 3.1. Consider the \(\text{GBTP}_1([2, 3], 6, 3 \times 4)\) below.

<table>
<thead>
<tr>
<th>{1,3}</th>
<th>{2,6}</th>
<th>{3,5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,2,3}</td>
<td>{2,5}</td>
<td>{3,4}</td>
</tr>
<tr>
<td>{4,5,6}</td>
<td>{3,6}</td>
<td>{1,5}</td>
</tr>
</tbody>
</table>

Each point \(x \in [6]\) gives a codeword \(c(x) = (r_1, r_2, \ldots, r_5)\), where \(r_j\) is the row in which point \(x\) appears in column \(j\). Hence, we have

\[
\begin{align*}
c(1) &= (2, 1, 3, 2), & c(2) &= (2, 2, 1, 3), & c(3) &= (2, 3, 2, 1), \\
c(4) &= (3, 1, 2, 3), & c(5) &= (3, 2, 3, 1), & c(6) &= (3, 3, 1, 2).
\end{align*}
\]

The code \(C = \{c(1), c(2), c(3), c(4), c(5), c(6)\}\) is an ESWC\((4, 3)_3\) of size 6.
Theorem 3.3 sets up the equivalence between GBTPs and ESWCs. In general, a GBTP may not correspond to an optimal ESWC. However, in the following, we look at specific $K$’s to derive families of optimal ESWCs.

### 3.2. Optimal ESWCs from GBTDs

A GBTP_\lambda (\{k\}; km, m \times \frac{\lambda (km-1)}{k-1}) is called a GBTD, denoted by GBTD_\lambda (k, m). In this case, we check that each pair of distinct points is contained in exactly \lambda blocks and every point is contained in either \left\lceil \frac{\lambda (km-1)}{m(k-1)} \right\rceil or \left\lfloor \frac{\lambda (km-1)}{m(k-1)} \right\rfloor cells of each row.

Applying Theorem 3.3, an ESWC(\lambda (km-1), \lambda(k-1)) of size km exists and the corresponding code is optimal by generalized Plotkin bound. So, we have the following.

**Theorem 3.4.** A GBTD_\lambda (k, m) exists if and only if an optimal ESWC(\lambda (km-1), \lambda(k-1)) of size km exists and attains the generalized Plotkin bound.

We remark that our definition of a GBTD extends that of Lamken and Vanstone [19], which corresponds in our definition to the case when \lambda = k - 1. The following summarizes the state-of-the-art results on the existence of GBTD_{k-1} (k,m).

**Theorem 3.5** (Lamken [19–22], Yin et al. [33], Chee et al. [9], Dai et al. [11]). The following holds.

(i) A GBTD_1 (2, m) exists if and only if m = 1 or m \geq 3.

(ii) A GBTD_2 (3, m) exists if and only if m = 1 or m \geq 3.

(iii) A GBTD_3 (4, m) exists if and only if m = 1 or m \geq 4.

(iv) A GBTD_4 (5, m) exists if m \geq 62 or m \in \{5 - 18, 30, 42, 46, 48 - 50, 54 - 57\}.

(v) A GBTD_{k-1} (k, k) exists if k is an odd prime power.

Theorem 2.3 (i)–(v) is now an immediate consequence of Theorems 3.4, 3.5, and Proposition 3.1. The existence of GBTD_\lambda (k, m) when \lambda \neq k - 1 has not been previously investigated. The smallest open case is when k = 3 and \lambda = 1, which is the case dealt with in this paper.

It follows, readily from the fact that a GBTD_1 (3, m) is also an RBIBD(3m, 3, 1), that a necessary condition for a GBTD_1 (3, m) to exist is that m must be odd. We note from Proposition 3.1 that A^{ESW}_{ESW} (4, 3) = 6 and A^{ESW}_{ESW} (7, 6) = 14, which do not meet the Plotkin bound. Hence, the corresponding designs GBTD_1 (3, 3) and GBTD_1 (3, 5) do not exist by Theorem 3.4.

Hence, a GBTD_1 (3, m) can exist only if m is odd and m \notin \{3, 5\}. In Sections 4–7, we prove that this necessary condition is also sufficient for the existence of GBTD_1 (3, m).

A direct consequence of this is Theorem 2.3 (vi).

### 3.3. Optimal ESWCs from GBTP_1 (\{2, 3^*\}; 2m + 1, m \times (2m - 3))

Theorem 3.4 constructs optimal ESWCs from GBTDs. In this subsection, we make slight variations to obtain another infinite family of optimal ESWCs.

Consider a GBTP_1 (\{2, 3\}; v, m \times n). If there is exactly one block of size 3 in each resolution class, then we denote the GBTP by GBTP_1 (\{2, 3^*\}; v, m \times n). A simple computation then shows v = 2m + 1. Now we establish the following construction for optimal ESWCs.
Theorem 3.6. Let $m \geq 7$. If there exists a $\text{GBTP}_1([2, 3^*]; 2m + 1, m \times (2m - 3))$, then there exists an optimal $\text{ESWC}(2m - 3, 2m - 4)_m$ of size $2m + 1$, which attains the generalized Plotkin bound.

Proof. By Theorem 3.3, we have a $\text{ESWC}(2m - 3, 2m - 4)_m$ of size $2m + 1$. It remains to verify its optimality.

Suppose otherwise that there exists an $\text{ESWC}(2m - 3, 2m - 4)_m$ of size $2m + 2$. Consider (1) in Theorem 2.2. On the left-hand side, we have

$$\binom{2m + 2}{2} \cdot (2m - 4) = 4m^3 - 2m^2 - 10m - 4.$$ 

Since $\left\lfloor \frac{2m+2+i}{m} \right\rfloor = 2$ for $0 \leq i \leq m - 3$ and $\left\lfloor \frac{2m+2+2(m-2)}{m} \right\rfloor = \left\lfloor \frac{2m+2+(m-1)}{m} \right\rfloor = 3$, the term on the right hand is

$$(2m - 3) \left( \sum_{i=0}^{m-3} 4(m - 3 - i) + 12 \right) + 9 = (2m - 3)(4m(m-2) - 2(m-3)(m-2) + 9) = 4m^3 - 2m^2 - 12m + 9.$$ 

But for $m \geq 7$,

$$4m^3 - 2m^2 - 10m - 4 > 4m^3 - 2m^2 - 12m + 9,$$

contradicting (1). Hence, an $\text{ESWC}(2m - 3, 2m - 4)_m$ of size $2m + 2$ does not exist and the result follows. $\square$

In the rest of this paper, we construct a $\text{GBTP}_1([2, 3^*]; 2m + 1, m \times (2m - 3))$ for $m \geq 4$, except possibly $m \in \{12, 13\}$. This with Theorem 3.6 and Proposition 3.1 gives Theorem 2.3 (vii).

4. PROOF STRATEGY OF THEOREM 2.3 (vi) AND THEOREM 2.3 (vii)

For the rest of the paper, we determine with finite possible exceptions the existence of $\text{GBTD}_1(3, m)$ and $\text{GBTP}_1([2, 3^*]; 2m + 1, m \times (2m - 3))$. Our proof is technical and rather complex. However, it follows the general strategy of the previous work [11, 21, 33]. This section outlines the general strategy used, and introduces some required combinatorial designs.

As with most combinatorial designs, using direct constructions to settle their existence is often difficult. Instead, we use recursive constructions, building big designs from smaller ones. Direct methods are used to construct a large enough set of small designs on which the recursions can work to generate all larger designs. For our recursive techniques to work, the GBTPs must possess more structure than stipulated in its definition. First, we consider $\text{GBTD}_1(3, m)$s that are $s$colorable, which are defined below.

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4.1. **c-**colorable GBTDS

We generalize the notion of factored GBTDS (FGTDs) introduced by Lamken [22]. FGTDS are crucial in the $k$-tupling construction for GBTDS of index $k-1$. However, when the index is 1, we extend this notion to $*$-colorability.

**Definition 4.1.** Let $c$ be positive. A $c$-colorable RBIBD($v$, $k$, $\lambda$) is an RBIBD($v$, $k$, $\lambda$) with the property that its $\frac{\lambda v(v-1)}{k(k-1)}$ blocks can be arranged in a $v \times \frac{\lambda v(v-1)}{k(k-1)}$ array, and each block can be colored with one of $c$ colors so that

(i) each point appears exactly once in each column, and
(ii) in each row, blocks of the same color are pairwise disjoint.

**Definition 4.2.** A GBTD($k$, $m$) is $c$-colorable if each of its blocks can be colored with one of $c$ colors so that in each row, blocks of the same color are pairwise disjoint.

**Definition 4.3.** A $k$-colorable RBIBD($v$, $k$, $1$) is $k$-colorable with property $\Pi$ if there exists a row $r$ such that for each color $i$, there exists a point (called a witness for $i$) that is not contained in any block in row $r$ that is colored $i$.

A GBTD($k$, $m$) that is $c$-colorable with property $\Pi$ is similarly defined.

**Example 4.1.** The RBIBD(15, 3, 1) in Fig. 1 is $3$-colorable with property $\Pi$.

**Proposition 4.4.** If an RBIBD($v$, $k$, 1) is $(k-1)$-colorable, then it is $k$-colorable with property $\Pi$.

**Proof.** Consider a $(k-1)$-colorable RBIBD($v$, $k$, 1) with colors $c_1, c_2, \ldots, c_{k-1}$. There must exist a point, say $x$, that appears only once in the first row. Recolor the block that contains this point with color $c_k$. This new coloring shows that the RBIBD($v$, $k$, 1) is $k$-colorable with property $\Pi$, since for the first row, the point $x$ is a witness for colors $c_1, c_2, \ldots, c_{k-1}$, and any point not in the block colored by $c_k$ is a witness for $c_k$. 

**Example 4.2.** The GBTD(3, 9) in Fig. 2 is $2$-colorable and is therefore $3$-colorable with property $\Pi$ by Proposition 4.4.

We note that a $3$-colorable RBIBD and a $3$-colorable RBIBD with property $\Pi$ are crucial in the tripling construction of a GBTD(1, $m$) and a special GBTD(1, (3, $m$), respectively (see Proposition 5.1). This is an adaptation of the $k$-tupling construction for GBTDS with index $k-1$ [22, Theorem 3.1]. However, we note certain differences. An FGTDS by definition is necessary a GBTD, while $*$-colorability and property $\Pi$ are

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*Journal of Combinatorial Designs* DOI 10.1002/jcd
where $A$ is the array

\[
\begin{array}{cccccc}
1_07_1\diamond 2_\Diamond & 6_03_2\diamond 1_\Diamond & 0_04_1\diamond 2_\Diamond & 6_07_1\diamond 0_2 & 7_00_1\diamond 2_\Diamond & 5_05_1\diamond 2_\Diamond & 1_14_2\diamond 0_0 \\
5_02_2\diamond 0_0 & 2_01_0\diamond 2_\Diamond & 7_04_2\diamond 1_\Diamond & 1_05_1\diamond 2_\Diamond & 4_06_1\diamond 2_\Diamond & 0_07_8\diamond 2_\Diamond & 7_5_1\diamond 0_2 \\
3_16_2\diamond 0_0 & 4_17_2\diamond 0_0 & 3_01_1\diamond 2_\Diamond & 0_52\diamond 0_1\diamond 2_\Diamond & 2_06_1\diamond 2_\Diamond & 3_07_1\diamond 2_\Diamond & 1_02_1\diamond 2_\Diamond \\
3_04_1\diamond 2_\Diamond & 1_01_1\diamond 2_\Diamond & 5_10_2\diamond 0_0 & 4_02_1\diamond 0_2 & 5_03\diamond 2_\Diamond & 2_07_2\diamond 1_\Diamond & 4_01_2\diamond 2_\Diamond \\
6_02_1\diamond 2_\Diamond & 4_05_1\diamond 2_\Diamond & 5_06_1\diamond 2_\Diamond & 6_12\diamond 0_0 & 1_06_2\diamond 1_\Diamond & 6_04\diamond 2_\Diamond & 3_03\diamond 0_2 \\
0_00_1\diamond 2_\Diamond & 5_02_1\diamond 2_\Diamond & 4_07_3\diamond 2_\Diamond & 2_01_1\diamond 2_\Diamond & 7_12\diamond 0_0 & 1_06_1\diamond 2_\Diamond & 0_03_7_2 \\
7_06_1\diamond 2_\Diamond & 7_03_1\diamond 2_\Diamond & 6_03_1\diamond 2_\Diamond & 5_00\diamond 0_1\diamond 2_\Diamond & 3_02_1\diamond 2_\Diamond & 0_13_2\diamond 0_0 & 2_07_1\diamond 2_\Diamond \\
2_05_1\diamond 2_\Diamond & 0_07_1\diamond 2_\Diamond & 2_02_1\diamond 2_\Diamond & 7_04_1\diamond 2_\Diamond & 6_01_1\diamond 2_\Diamond & 4_03_1\diamond 2_\Diamond & 6_06\diamond 1_2 \\
4_01_1\diamond 2_\Diamond & 3_06_1\diamond 2_\Diamond & 1_00_5_2 & 3_03_1\diamond 2_\Diamond & 0_05_1\diamond 2_\Diamond & 7_02_1\diamond 2_\Diamond & 5_04_1\diamond 2_\Diamond \\
\end{array}
\]

where $B$ is the array

\[
\begin{array}{cccccc}
0_06_1\diamond 2_\Diamond & 4_06_1\diamond 2_\Diamond & 1_02_0\diamond 4_\Diamond & 2_03_0\diamond 5_0 & 2_27_2\diamond 0_2 & 2_13_1\diamond 5_1 \\
2_15_2\diamond 0_0 & 5_07_1\diamond 2_\Diamond & 3_14_1\diamond 6_1 & 0_11_3_1 & 3_04_6_0 & 3_20_2_1_2 \\
2_03_1_2 & 6_0\diamond 7_2 & 4_2_1_2\diamond 2_\Diamond & 4_15_1_7_1 & 5_16_1_0_1 & 4_05_7_0 \\
7_07_1_7 & 0_02_1_2 & 0_2_5_2_6_2 & 6_0_7_0_1_0 & 6_2_3_2_4_2 & 6_17_1_1 \\
5_01_3_2 & 1_03_1_2 & 7_1_0_2_1 & 5_2_2_3_2 & 7_0_0_0_2 & 7_2_4_5_2 \\
6_05_1_2_2 & 2_0_4_1_3_2 & 3_0_7_0_\diamond 0 & 1_2_6_2_7_2 & 5_2_2_3_2 & 4_0_1_0_1 \diamond \\
1_04_1_0_2 & 3_0_5_1_4_2 & 1_1_5_1_\diamond 1_1 & 4_0_0_0_\diamond 0_0 & 1_2_1_4_1 & 6_2_3_2_\diamond 2_2 \\
3_0_0_1_6_2 & 1_1_0_0_\diamond 1_1 & 3_2_7_2_\diamond 2_2 & 2_1_6_1_1 & 3_0_1_0_0_0 & 0_0_1_0_0_0 \diamond \\
4_0_1_2_1_0_2 & 7_0_1_1_0_2 & 5_0_6_0_0_0 & 4_2_0_2_\diamond 0_2 & 3_1_7_1_0_1 & 6_0_2_0_0_0 \diamond
\end{array}
\]

**FIGURE 2.** A 2-\(\ast\)-colorable special GBTD\(_1\)(3, 9) \((X, A)\), where \(X = (\mathbb{Z}_6 \times \mathbb{Z}_3) \cup \{\infty, 0_\Diamond, 1_\Diamond, 2_\Diamond\}\) and colors \(\{\blacklozenge, \Diamond\}\). The cell \((1, 5)\), occupied by the block \(7_00_1\_2\), is special. For succinctness, a set \(\{x, y, z\}\) is written \(xyz\).

defined for RBIBDs. Hence, we do not need a smaller GBTD to seed the recursion in Proposition 5.1. We make use of this fact to yield a special GBTD\(_1\)(3, 15) in Lemma 7.1.

### 4.2. Incomplete GBTPs

Incomplete designs are ubiquitous in combinatorial design theory and crucial in “filling in the holes” constructions described in Section 5.

Suppose that \((X, A)\) is a \((v, K, \lambda)\)-packing. Let \(W \subset X\) with \(|W| = w\). Furthermore, we call \((X, W, A)\) as an *incomplete resolvable packing*, denoted by \(\text{IRP}(v, K, \lambda; w)\), if it satisfies the following conditions:

(i) any pair of points from \(W\) occurs in no blocks of \(A\),

(ii) the blocks in \(A\) can be partitioned into parallel classes and *partial parallel classes* \(X \setminus W\).
Definition 4.5. Let \((X, W, A)\) be an IRP\((v, K, \lambda; w)\). Then \((X, W, A)\) is called an incomplete GBTP (IGBTP) if the blocks of \(A\) are arranged into an \(m \times n\) array \(A\), with rows and columns indexed by \(R\) and \(C\), respectively, satisfying the following conditions:

(i) there exist a \(P \subset R\) with \(|P| = m'\) and a \(Q \subset C\) with \(|Q| = n'\) such that the cell \((r, c)\) is empty if \(r \in P\) and \(c \in Q\);
(ii) for any row \(r \in P\), every point in \(X \setminus W\) is contained in either \(\lceil n/m \rceil\) or \(\lfloor n/m \rfloor\) cells and the points in \(W\) do not appear; for any row \(r \in R \setminus P\), every point in \(X\) is contained in either \(\lceil n/m \rceil\) or \(\lfloor n/m \rfloor\) cells;
(iii) the blocks in any column \(c \in Q\) form a partial parallel class of \(X \setminus W\) and the blocks in any column \(c \in C \setminus Q\) forms a parallel class of \(X\).

Denote such an IGBTP by IGBTP\(_{\lambda}(K, v, m \times n; w, m' \times n')\).

Example 4.3. An IGBTP\(_1([2, 3^*], 29, 14 \times 25; 9, 4 \times 5)\) is given in Fig. 3.

Consider an IGBTP\(_1([k], km, m \times \frac{km-1}{k-1}; k, 1 \times 1)\). Then its corresponding array has one empty cell and we fill this cell with the block \(W\) to obtain a GBTD\(_1(k, m)\). A GBTD\(_1(k, m)\) obtained in this way is called a special GBTD\(_1(k, m)\) and the cell occupied by \(W\) is said to be special.

Example 4.4. The GBTD\(_1(3, 9)\) in Fig. 2 is a special GBTD\(_1(3, 9)\) with special cell \((1, 5)\).

A few more classes of auxiliary designs are also required.

4.3. Group Divisible Designs and Transversal Designs

Definition 4.6. Let \((X, A)\) be a set system and let \(G = \{G_1, G_2, \ldots, G_s\}\) be a partition of \(X\) into subsets, called groups. The triple \((X, G, A)\) is a group divisible design (GDD) when every 2-subset of \(X\) not contained in a group appears in exactly one block, and \(|A \cap G| \leq 1\) for \(A \in A\) and \(G \in G\).

We denote a GDD \((X, G, A)\) by \(K\)-GDD if \((X, A)\) is \(K\)-uniform. The type of a GDD \((X, G, A)\) is the multiset \(\langle|G| : G \in G\rangle\). For convenience, the exponential notation is used to describe the type of a GDD: a GDD of type \(g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}\) is a GDD with exactly \(t_i\) groups of size \(g_i, i \in [s]\).

Definition 4.7. A transversal design TD\((k, n)\) is a \(\{k\}\)-GDD of type \(n^k\).

The following result on the existence of transversal designs (see [1]) is sometimes used without explicit reference throughout this paper.

Theorem 4.8. Let TD\((k)\) denote the set of positive integers \(n\) such that there exists a TD\((k, n)\). Then, we have

\[(i)\quad \text{TD}(4) \supseteq \mathbb{Z}_{>0} \setminus \{2, 6\},\]
\[(ii)\quad \text{TD}(5) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 6, 10\},\]
\[(iii)\quad \text{TD}(6) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 6, 10, 14, 18, 22\},\]
\[(iv)\quad \text{TD}(7) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\},\]
\[(v)\quad \text{TD}(k) \supseteq \{q : q \geq k - 1 \text{ is a prime power}\}.\]
where $A$ is the array

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**FIGURE 3.** An 1GBTP$_1([2,3],29,14 \times 25;9,4 \times 5)$ $(X,A)$, where $X = \mathbb{Z}_{20} \cup \{a,b,c,d,e,f,g,h,i\}$ and $W = \{a,b,c,d,e,f,g,h,i\}$. For succinctness, a block $\{x,y,z\}$ is written $x,y,z$.

**Definition 4.9.** A doubly resolvable TD($k,n$), denoted by DRTD($k,n$), is a TD($k,n$) whose blocks can be arranged in an $n \times n$ array such that each point appears exactly once in each row and once in each column.

The following proposition describes the relationship between DRTDs and TDs.

**Proposition 4.10** (Folklore, see [1, Theorem 3.18] and [10]). There exists a TD($k+2,n$) if and only if there exists a DRTD($k,n$).
Corollary 4.11. A DRTD\((3, n)\) exists for all \(n \geq 4\) and \(n \not\in \{6, 10\}\).

\(\Box\)

4.4. Frame GBTD

Let \((X, \mathcal{G}, A)\) be a \([k]\)-GDD with \(\mathcal{G} = \{G_1, G_2, \ldots, G_s\}\) and \(|G_i| \equiv 0 \pmod{k(k-1)}\) for all \(i \in [s]\). Let \(R = \frac{1}{k} \sum_{i=1}^{s} |G_i|\) and \(C = \frac{1}{k-1} \sum_{i=1}^{s} |G_i|\). Suppose there exists a partition \([R] = \bigsqcup_{i=1}^{s} R_i\) and a partition \([C] = \bigsqcup_{i=1}^{s} C_i\) such that for each \(i \in [s]\), we have \(|R_i| = |G_i|/k\) and \(|C_i| = |G_i|/(k-1)\).

We say that \((X, \mathcal{G}, A)\) is a frame GBTD (FrGBTD) if its blocks can be arranged in an \(R \times C\) array such that the following conditions hold:

1. the cell \((r, c)\) is empty when \((r, c) \in R_i \times C_i\) for \(i \in [s]\),
2. for any row \(r \in R_i\), each point in \(X \setminus G_i\) appears either once or twice and the points in \(G_i\) do not appear,
3. for any column \(c \in C_i\), each point in \(X \setminus G_i\) appears exactly once.

Denote this FrGBTD by FrGBTD\((k, T)\), where \(T = \langle |G_i| : i \in [s] \rangle\).

Example 4.5. An FrGBTD\((3, 6^6)\) is given in Fig. 4.

5. RECURSIVE CONSTRUCTIONS

In this section, we describe the necessary recursive constructions. We note that these are straightforward adaptations of methods in previous work [11, 21, 22, 33]. Here, we state the propositions without proof and the interested reader may refer to [6] for detailed proofs.

5.1. Recursive Constructions for GBTPs

First, for block size 3, we have the following tripling construction for GBTDs. This is an adaption of \(k\)-tupling construction for the case of GBTDs with index \(k-1\) [22, Theorem 3.1] and the doubling construction for balanced tournament designs [27].

Proposition 5.1 (Tripling construction). Suppose there exists a 3-\(*\)colorable RBIBD\((m, 3, 1)\) and a DRTD\((3, m)\). Then there exists a 2-\(*\)colorable GBTD\(_1\)(3, \(m\)). Suppose further that the RBIBD\((m, 3, 1)\) is 3-\(*\)colorable with property \(\Pi\). Then the GBTD\(_1\)(3, \(m\)) is a special GBTD\(_1\)(3, \(m\)).

Corollary 5.2. Let \(m > 3\) and suppose an RBIBD\((m, 3, 1)\) that is 3-\(*\)colorable with property \(\Pi\) exists. Then there exists a special GBTD\(_1\)(3, \(3^km\)) for all \(k \geq 0\).

\(\Box\)

The following propositions are simple generalizations of the standard “filling in the hole” construction to construct GBTPs or GBTDs using IGBTPs and FrGBTDs.
where \(A\) is the array

\[
\begin{array}{cccccccccc}
- & - & - & 4_0 & 1_0 & 7_0 & 4_1 & 1_7 & 1_1 & 4_2 \\
- & - & - & 6_0 & 7_2 & 8_0 & 6_1 & 7_0 & 8_1 & 4_2 \\
2_0 & 8_1 & 1_0 & 2_1 & 8_2 & 1_2 & 2_2 & 8_0 & 1_2 & 4_1 \\
6_2 & 7_0 & 3_1 & 6_0 & 7_0 & 3_2 & 6_1 & 7_0 & 3_0 & 4_2 \\
4_0 & 0_1 & 3_0 & 4_1 & 0_3 & 1_2 & 4_2 & 0_3 & 2_2 & 1_0 \\
8_2 & 9_1 & 5_1 & 8_0 & 0_9 & 5_2 & 8_1 & 9_1 & 5_2 & 4_1 \\
6_0 & 2_1 & 5_0 & 6_1 & 2_5 & 1_1 & 6_2 & 0_5 & 2_8 & 1_1 \\
0_2 & 1_7 & 0_1 & 0_1 & 7_0 & 2_0 & 3_1 & 1_2 & 1_3 & 2_3 \\
8_0 & 4_1 & 7_0 & 8_1 & 4_2 & 7_1 & 8_2 & 4_0 & 7_2 & 4_2 \\
2_3 & 2_9 & 1_2 & 2_3 & 9_0 & 2_1 & 3_1 & 9_0 & 2_4 & 3_2 \\
0_0 & 6_9 & 0_1 & 0_2 & 9_0 & 2_6 & 9_2 & 2_6 & 9_0 & 4_1 \\
4_2 & 5_1 & 1_2 & 4_0 & 5_1 & 1_2 & 4_1 & 5_0 & 1_2 & 4_1 \\
\end{array}
\]

where \(B\) is the array

\[
\begin{array}{cccccccccc}
8_0 & 0_0 & 1_0 & 8_1 & 5_1 & 1_1 & 8_2 & 5_2 & 1_2 & 0_0 \\
0_1 & 2_0 & 0_0 & 1_0 & 2_1 & 2_0 & 1_2 & 0_0 & 2_3 & 0_0 \\
6_2 & 3_0 & 0_2 & 6_1 & 3_0 & 0_2 & 6_2 & 3_0 & 0_0 & 1_0 \\
7_2 & 8_1 & 3_2 & 7_0 & 8_2 & 3_2 & 7_1 & 8_0 & 3_1 & 7_2 \\
6_1 & 0_2 & 0_4 & 6_0 & 0_2 & 0_3 & 6_2 & 5_0 & 0_2 & 6_1 \\
1_3 & 2_5 & 0_5 & 1_0 & 3_0 & 1_2 & 1_3 & 0_3 & 2_0 & 0_1 \\
- & - & - & 8_1 & 2_2 & 4_3 & 8_2 & 2_0 & 0_0 & 1_2 \\
- & - & - & 3_5 & 2_0 & 0_5 & 3_2 & 5_0 & 2_0 & 0_2 \\
9_1 & 7_0 & 8_1 & 5_0 & 7_1 & 3_2 & 5_0 & 3_0 & 7_2 & 4_0 \\
8_2 & 0_2 & 2_3 & 8_1 & 0_2 & 2_0 & 5_0 & 1_2 & 0_0 & 1_2 \\
2_1 & 5_0 & 0_0 & 2_0 & 5_0 & 1_2 & 7_1 & 0_0 & 5_1 & 2_1 \\
6_0 & 7_0 & 1_2 & 6_1 & 7_2 & 0_0 & 6_2 & 7_0 & 1_0 & 6_0 \\
\end{array}
\]

**FIGURE 4.** An FrGBTD\(_1\)(3, 6\(^n\)) \((X, G, A)\), where \(X = (\mathbb{Z}_{10} \times \mathbb{Z}_3) \cup \\{\infty_i : i \in \mathbb{Z}_6\}\) and \(G = \{t_0, t_1, t_2, (5 + t_0), (5 + t_1), (5 + t_2) : t \in \mathbb{Z}_5\} \cup \{\infty_i : i \in \mathbb{Z}_6\}\). For succinctness, a set \(\{x, y, z\}\) is written \(xyz\).

**Proposition 5.3 (IGBP construction for GBTP).** If an \(\text{IGBTP}_\lambda(K, v, m \times n; w, m' \times n')\) and a \(\text{GBTP}_\lambda(K, w, m' \times n')\) exists, then a \(\text{GBTP}_\lambda(K, v, m \times n)\) exists.

FrGBTD is a useful tool to construct larger GBTPs from smaller ones.

**Proposition 5.4 (FrGBTD construction for GBTP).** Let \(k \in K\). Suppose there exists an FrGBTD\(_k\)(T)(X, G, A), where \(G = \{G_1, G_2, \ldots, G_s\}\), and let \(r_i = \frac{|G_i|}{k}\) and \(c_i = \frac{|G_i|}{(k - 1)}\), for \(i \in [s]\). If there exists an IGBTP\(_i\)(K, \(|G_i| + w, (r_i + m) \times (c_i + n); w, m \times n\) for all \(i \in [s]\), then there exists an IGBTP\(_i\)(K, \(\sum_{i=1}^{s} |G_i| + w, (\sum_{i=1}^{s} r_i + \sum_{i=1}^{s} c_i) \times (\sum_{i=1}^{s} m_i + \sum_{i=1}^{s} n_i)\) w, m \times n).
Furthermore, if a GBTP \(_1(K, w, m \times n)\) exists, then a GBTP \(_1(K, \sum_{i=1}^s |G_i| + w, (\sum_{i=1}^s r_i + m) \times (\sum_{i=1}^s c_i + n))\) exists.

Since a GBTD is an instance of GBTP, we have the following recursive construction for GBTDs.

**Corollary 5.5** (FrGBTD construction for GBTD). Suppose an FrGBTD\((k, T)\) exists with groups \(\{G_1, G_2, \ldots, G_s\}\). Let \(g_i = |G_i|/k\), for \(i \in [s]\). If there exists a special GBTD\(_1(k, g_i + 1)\) for all \(i \in [s]\), then there exists a special GBTD\(_1(k, \sum_{i=1}^s g_i + 1)\).

When the groups are of the same size, we have the following corollary.

**Corollary 5.6.** If there exists an FrGBTD\((3, (3g_t)^t)\) and a special GBTD\(_1(3, g + 1)\), then there exists a special GBTD\(_1(3, gt + 1)\).

For Proposition 5.3 and Corollary 5.5 to be useful, we require large classes of FrGBTDs. We give three recursive constructions for FrGBTDs next.

### 5.2. Recursive Constructions for FrGBTDs

We adapt the standard direct product construction.

**Proposition 5.7** (Inflation). Suppose an FrGBTD\((k, T)\) and a DRTD\((k, n)\) exists. Then there exists an FrGBTD\((k, nT)\).

Wilson’s fundamental construction for GDDs [32] can also be modified to construct FrGBTDs.

**Proposition 5.8** (Fundamental construction). Suppose there exists a (master) GDD \((X, G, A)\) of type \(T\) and let \(w : X \rightarrow \mathbb{Z}_{\geq 0}\) be a weight function. If for each \(A \in A\), there exists an (ingredient) FrGBTD\((k, \langle w(a) : a \in A\rangle)\), then there exists an FrGBTD\((k, (\sum_{x \in G} w(x) : G \in G))\).

Proposition 5.8 admits the following specialization.

**Proposition 5.9** FrGBTD from truncated TD. Let \(s > 0\). Suppose there exists a TDI\((u + s, m)\), and \(g_1, g_2, \ldots, g_s\) are non-negative integers at most \(m\). If there exists an FrGBTD\(_1(k, g^t)\) for each \(t \in \{u, u + 1, \ldots, u + s\}\), then there exists an FrGBTD\(_1(k, T)\), where \(T = (g \cdot m)^u (g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_s)\).

### 6. DIRECT CONSTRUCTIONS

This section constructs some small GBTDs and FrGBTDs that are required to seed the recursive constructions given in Section 5. Our main tools are starters and the method of differences.

Starter–adder constructions are ubiquitous in the constructions for GBTDs with index \(k - 1\), associated frames, and other types of similar designs (see, e.g., [9, 11, 21, 22, 33]). Unlike previous work and due to the lack of symmetry in our arrays, we fix the positions of the starters in our arrays and “develop” the blocks in a variety of “directions” (see Figs. 5–7). This removes the use of adders and surprisingly a careful analysis of the starter conditions allows a prime power construction that is given in Proposition 6.3.
where $A$ is the array

\[
\begin{array}{ccccccc}
A_0 & A_{-\alpha_1} + \alpha_1 & A_{-\alpha_2} + \alpha_2 & \cdots & A_{-\alpha_{m-1}} + \alpha_{m-1} \\
A_{\alpha_1} & A_0 + \alpha_1 & A_{\alpha_1-\alpha_2} + \alpha_2 & \cdots & A_{\alpha_1-\alpha_{m-1}} + \alpha_{m-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
A_{\alpha_{m-1}} & A_{\alpha_{m-1}-\alpha_1} + \alpha_1 & A_{\alpha_{m-1}-\alpha_2} + \alpha_2 & \cdots & A_0 + \alpha_{m-1}
\end{array}
\]

and $B$ is the array

\[
\begin{array}{ccccccc}
B_1 & B_2 & \cdots & B_{(m-1)/(k-1)} \\
B_1 + \alpha_1 & B_2 + \alpha_1 & \cdots & B_{(m-1)/(k-1)} + \alpha_1 \\
\vdots & \vdots & \ddots & \vdots \\
B_1 + \alpha_{m-1} & B_2 + \alpha_{m-1} & \cdots & B_{(m-1)/(k-1)} + \alpha_{m-1}
\end{array}
\]

**FIGURE 5.** A GBTD$_1(k, m)$ from $(\Gamma \times [k])$-GBTD-starter $S = \{A_\alpha : \alpha \in \Gamma \} \cup \{B_t : t \in T\}$, where $\Gamma = \{0, \alpha_1, \ldots, \alpha_{m-1}\}$ and $T = [(m-1)/(k-1)]$.

First, we recall certain concepts with regards to the method of differences. Let $\Gamma$ be an additive abelian group and let $n$ be a positive integer. For a set system $(\Gamma, S)$, the difference list of $S$ is the multiset

$$\Delta S = \langle x - y : x, y \in A, x \neq y, \text{ and } A \in S \rangle.$$ 

For a set system $(\Gamma \times [n], S)$ and $i, j \in [n]$, the multiset

$$\Delta_{ij} S = \langle x - y : x_i, y_j \in A, x_i \neq y_j, \text{ and } A \in S \rangle$$

is called a list of pure differences when $i = j$, and called a list of mixed differences when $i \neq j$.

### 6.1. Direct Constructions for GBTDs

**Definition 6.1** (Starter for GBTD). Let $m$ be an odd positive integer, $\Gamma$ be an additive abelian group of size $m$. Let $T$ be an index set of size $(m - 1)/2$. Let $(\Gamma \times [3], S)$ be a $\{3\}$-uniform set system of size $(3m - 1)/2$, where

$$S = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}.$$ 

$S$ is called a $(\Gamma \times [3])$-GBTD-starter if the following conditions hold:

(i) $\Delta_{ii} S = \Gamma \setminus \{0\}$, for $i \in [3]$,
(ii) $\Delta_{ij} S = \Gamma$, for $i, j \in [3], i \neq j$,
(iii) $\bigcup_{\alpha \in \Gamma} A_\alpha = \Gamma \times [3]$.
where $W$ is a $(w-1)/2 \times (w-4)$ empty array, $A$ is an $m \times (w-4)$ array,

$$
\begin{array}{cccc}
\{0,0\} & A_1 & A_1 + 0_1 & A_2 + 0_1 \\
\{1,1\} & A_1 + 1_0 & A_1 + 1_1 & A_2 + 1_1 \\
& \vdots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & A_1 - 1_0 & A_1 - 1_1 & A_2 - 1_1 \\
\end{array}
$$

$B$ and $C$ are the following $(w-1)/2 \times m$ and $m \times m$ arrays,

$$
\begin{array}{cccc}
B_1 & B_1 + 1_0 & \cdots & B_1 - 1_0 \\
B_2 & B_1 + 1_0 & \cdots & B_1 - 1_0 \\
& \vdots & \ddots & \vdots \\
B_{(w-1)/2} & B_{(w-1)/2} + 1_0 & \cdots & B_{(w-1)/2} - 1_0 \\
\end{array},
\begin{array}{cccc}
C_0 & C_{m-1} + 1_0 & \cdots & C_{1} - 1_0 \\
C_1 & C_0 + 1_0 & \cdots & C_2 - 1_0 \\
& \vdots & \ddots & \vdots \\
C_{m-1} & C_{m-2} + 1_0 & \cdots & C_0 - 1_0 \\
\end{array}
$$

**FIGURE 6.** An IGBTP$_1([2, 3^*], 2m + w, (m + (w - 1)/2) \times (2m + w - 4); w, (w - 1)/2 \times (w - 4))$ from a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-GBTP-starter.

where $W$ is a $4 \times 5$ empty array, $A$ is a $2m \times 5$ array,

$$
\begin{array}{cccc}
\{0,0\} & \{x_0, x_2\} & \{y_0, y_3\} & A_1 \\
\{1,1\} & \{x_1, x_3\} & \{y_1, y_2\} & A_1 + 1_0 \\
& \vdots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & \{(x-1)_0, x_2\} & \{(y-1)_0, (y+1)_3\} & A + (m-1)_0 \\
\end{array},
\begin{array}{cccc}
\{0,0\} & \{0_0, 0_1\} & \{0_2, 0_3\} & A_1 \\
\{1,1\} & \{1_0, 1_1\} & \{1_2, 1_3\} & A + 1_1 \\
& \vdots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & \{(x-1)_0, x_2\} & \{(y-1)_0, (y+1)_3\} & A + (m-1)_1 \\
\end{array}
$$

$B$, $C$ and $D$ are the following $4 \times m$, $m \times m$ and $m \times m$ arrays respectively,

$$
\begin{array}{cccc}
B_1 & B_1 + 1_0 & \cdots & B_1 - 1_0 \\
B_2 & B_2 + 1_0 & \cdots & B_2 - 1_0 \\
B_3 & B_3 + 1_0 & \cdots & B_3 - 1_0 \\
B_4 & B_4 + 1_0 & \cdots & B_4 - 1_0 \\
\end{array},
\begin{array}{cccc}
C_0 & C_{m-1} + 1_0 & \cdots & C_1 - 1_0 \\
C_1 & C_0 + 1_0 & \cdots & C_2 - 1_0 \\
& \vdots & \ddots & \vdots \\
C_{m-1} & C_{m-2} + 1_0 & \cdots & C_0 - 1_0 \\
\end{array},
\begin{array}{cccc}
D_0 & D_{m-1} + 1_0 & \cdots & D_1 - 1_0 \\
D_1 & D_0 + 1_0 & \cdots & D_2 - 1_0 \\
& \vdots & \ddots & \vdots \\
D_{m-1} & D_{m-2} + 1_0 & \cdots & D_0 - 1_0 \\
\end{array}
$$

**FIGURE 7.** An IGBTP$_1([2, 3^*], 4m + 9, (2m + 4) \times (4m + 5); 9, 4 \times 5)$ from a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-GBTP-starter.
(iv) \( \{ j : \alpha_j \in B_t \text{ for some } \alpha \in \Gamma \} = [3] \), for \( t \in T \),
(v) each element in \( \Gamma \times [3] \) appears either once or twice in the multiset

\[
R = \left( \bigcup_{\alpha \in \Gamma} A_\alpha - \alpha \right) \cup \left( \bigcup_{t \in T} B_t \right).
\]

Furthermore, \( S \) is said to be special if
(vi) each element in \( A_0 \) appears exactly once in \( R \).

Also, \( S \) is said to be 3-\( * \)-colorable with property \( \Pi \) if each of the blocks in

\[
\{ A_\alpha - \alpha : \alpha \in \Gamma \} \text{ and } \{ B_t : t \in T \},
\]

can be colored with one of three colors so that
(vii) blocks of the same color are pairwise disjoint,
(viii) for each color \( c \), there exists a point (a witness for \( c \)) that is not contained in any block assigned color \( c \).

**Proposition 6.2.** If a \((\Gamma \times [k])\)-GBTD-starter exists, then a GBTD\(_1\)(\(k, m\)) exists. Similarly, if there exists a special \((\Gamma \times [3])\)-GBTD-starter, then there exists a special GBTD\(_1\)(3, \( m\)); and if there exists a 3-\( * \)-colorable \((\Gamma \times [3])\)-GBTD-starter with property \( \Pi \), then there exists a 3-\( * \)-colorable GBTD\(_1\)(3, \( m\)) with property \( \Pi \).

**Proof.** Let \( X = \Gamma \times [k] \), and suppose \( S = \{ A_\alpha : \alpha \in \Gamma \} \cup \{ B_t : t \in T \} \) is an \((\Gamma \times [k])\)-GBTD-starter. Let

\[
\mathcal{A} = \bigcup_{A \in S} \{ A + \alpha : \alpha \in \Gamma \}.
\]

Then \((X, \mathcal{A})\) is a BIBD\((km, k, 1)\), whose blocks can be arranged in an \( m \times \frac{(km-1)}{k-1} \) array, whose rows and columns are indexed by \( \Gamma \) and \( \Gamma \cup T \), respectively, as follows:

- for \( \alpha, \beta \in \Gamma \), the block \( A_\alpha + \beta \) is placed in cell \((\alpha + \beta, \beta)\), and
- for \( t \in T \) and \( \alpha \in \Gamma \), the block \( B_t + \alpha \) is placed in cell \((\alpha, t)\).

Figure 5 depicts the placement of blocks in the array.

For \( \beta \in \Gamma \), the set of blocks occupying column \( \beta \) is \( \{ A_\alpha + \beta : \alpha \in \Gamma \} \), which form a resolution class by condition (iii) of Definition 6.1. Similarly, for \( t \in T \), the set of blocks occupying column \( t \) is \( \{ B_t + \alpha : \alpha \in \Gamma \} \), which form a resolution class by condition (iv) in Definition 6.1.

The set of blocks occupying row 0 is given by \( R \), and by condition (v) of Definition 6.1, each point in \( X \) appears either once or twice in row 0. Since the blocks occupying row \( \alpha \) (\( \alpha \in \Gamma \)) are exactly the translates of the blocks in \( R \) by \( \alpha \), every point in \( X \) also appears either once or twice in row \( \alpha \).

Suppose \( S = \{ A_\alpha : \alpha \in \Gamma \} \cup \{ B_t : t \in T \} \) is a special \((\Gamma \times [3])\)-GBTD-starter. Then condition (vi) of Definition 6.1 ensures that the cell \((0, 0)\) is special.

On the other hand, if \( S \) be a 3-\( * \)-colorable \((\Gamma \times [3])\)-GBTD-starter and let

\[
c_i \text{ be the color assigned to } \begin{cases} A_i - i, & \text{if } i \in \Gamma, \\ B_i, & \text{otherwise.} \end{cases}
\]
For \( \alpha, \beta \in \Gamma \) and \( t \in T \), assign the block \( A_\alpha + \beta \) color \( c_\alpha \) and the block \( B_t + \beta \) color \( c_t \). Then conditions (vii) and (viii) of Definition 6.1 ensure that the GBTD\(_1(3, m)\) is 3-\(*\)colorable with property \( \Pi \). \( \square \)

**Proposition 6.3.** Let \( q \equiv 1 \mod 6 \). Then there exists a special \((\mathbb{F}_q \times [3])-\)GBTD-starter that is 3-\(*\)colorable with property \( \Pi \).

**Proof.** Let \( s = (q - 1)/6 \) and \( \omega \) be a primitive element of \( \mathbb{F}_q \). Consider \( \gamma \in \mathbb{F}_q \) that satisfies the following conditions (note that \( \omega^{2s} \) has order three):

(A) \( \gamma \not\in \{0, -1, -\omega^{2s}, -\omega^{4s}\} \);

(B) \( \gamma \not\in \{\omega^{2s} - \omega^{2js}, \omega^s - \omega^{2js} : i \neq j \in [3], t \in [s - 1]\} \).

The existence of \( \gamma \) is guaranteed since the cardinality of the union of sets in (A) and (B) is at most \( 4 + 6(s - 1) < 6s + 1 = q \).

Define \( \Lambda \) to be \( \{-\gamma \omega^{t-1 + 2(j-1)s} : t \in [s], j \in [3]\} \) and construct the following \( q + 3s = (3q - 1)/2 \) blocks. For \( \alpha \in \mathbb{F}_q \), let

\[
A_\alpha = \left\{ \left( \omega^{s - 1 + 2(j - 1)s} \right), \left( -\gamma \omega^{s - 2(i - 1)s} \right) : j \in [3] \right\}, \quad \text{if} \quad \alpha = -\gamma \omega^{t-1 + 2(i-1)s} \text{ where } t \in [s], i \in [3],
\]

\[
\text{otherwise.}
\]

For \( (t, j) \in [s] \times [3] \), let

\[
B_{(t, j)} = \{(\omega^{s - 1 + 2(j - 1)s}(\omega^{2(i - 1)s} + \gamma)) : i \in [3]\}.
\]

Let \( S = \{A_\alpha : \alpha \in \mathbb{F}_q \} \cup \{B_{(t, j)} : (t, j) \in [s] \times [3]\} \) and we claim that \( S \) is the desired starter.

Define

\[
D = \{(\omega^{s - 1 + 2(j - 1)s} : j \in [3]) : t \in [s]\},
\]

and Wilson [31] showed that the blocks in \( D \) are mutually disjoint and \( \Delta D = \mathbb{F}_q \setminus \{0\} \).

Hence, for condition (i) of Definition 6.1, we check for \( i \in [3], \)

\[
\Delta_{ii} S = \Delta_{ii} \{A_\alpha : \alpha = -\gamma \omega^{t-1 + 2(i-1)s}, t \in [s], i \in [3]\}
\]

\[
\Delta_{ii} D = \mathbb{F}_q \setminus \{0\}.
\]

For condition (ii), we verify for \( i \neq i' \in [3], \)

\[
\begin{align*}
\Delta_{ii'} S & = \bigcup_{\alpha \in \Lambda} \left( -\frac{\alpha}{\gamma}(\omega^{2(i - 1)s} - \omega^{2(i' - 1)s}) \right) \cup \bigcup_{(t, j) \in [s] \times [3]} \omega^{s - 1 + 2(j - 1)s}(\omega^{2(i - 1)s} - \omega^{2(i' - 1)s}) \\
& = (\omega^{2(i - 1)s} - \omega^{2(i' - 1)s}) \left( \bigcup_{\alpha \in \Lambda} -\frac{\alpha}{\gamma} \cup \bigcup_{(t, j) \in [s] \times [3]} \omega^{s - 1 + 2(j - 1)s} \right) \\
& = (\omega^{2(i - 1)s} - \omega^{2(i' - 1)s}) \mathbb{F}_q = \mathbb{F}_q.
\end{align*}
\]

For condition (iii) of Definition 6.1, since the number of points in \( \bigcup_{\alpha \in \mathbb{F}_q} A_\alpha \) is \( kq \), it suffices to check that each point \( \beta_i \in \mathbb{F}_q \times [k] \) belongs to some block \( A_\alpha \). Indeed, if \( \beta_i/\omega^{2(i - 1)s} = \omega^{2(t - 1 + 2(j - 1)s)} \) for some \( (t, j) \in [s] \times [3] \), then let \( \alpha = -\gamma \omega^{t-1 + 2(i-1)s} \).
and so, \( \beta_i = (\omega^{i-1+2(i+j-3)^2})_i \) belongs to \( A_\alpha \). Otherwise, \(-\gamma \beta / \omega^{2(i-1)s} \notin \Lambda \). Let \( \alpha = -\gamma \beta / \omega^{2(i-1)s} \) and \( \beta_i \in A_\alpha \) as desired.

Condition (iv) of Definition 6.1 is clearly true from the definition of \( B_{(t,j)} \). We establish condition (v) of Definition 6.1 through the following claims:

**Claim 6.1.** The blocks in \( \bigcup_{\alpha \notin \Lambda} (A_\alpha - \alpha) \cup \bigcup_{(t,j) \in [s] \times [3]} B_{(t,j)} \) form a resolution class.

As above, it suffices to check that each point \( \beta_i \in \mathbb{F}_q \times [3] \) belongs to some block in \( \bigcup_{\alpha \notin \Lambda} (A_\alpha - \alpha) \cup \bigcup_{(t,j) \in [s] \times [k]} B_{(t,j)} \) as the total number of points is \( kq \).

Indeed, if \( \beta / (\omega^{2(i-1)s} + \gamma) = \omega^{i-1+2(j-1)s} \) for some \( (t, j) \in [s] \times [k] \), then \( \beta_i \in B_{(t,j)} \). Otherwise, \(-\gamma \beta / (\omega^{2(i-1)s} + \gamma) \notin \Lambda \). Let \( \alpha = -\gamma \beta / (\omega^{2(i-1)s} + \gamma) \) (note that \( \alpha \) is well defined by Condition (A)) and \( \beta_i \in A_\alpha - \alpha \).

**Claim 6.5.** Each point in \( \mathbb{F}_q \times [k] \) appears at most once in \( \bigcup_{\alpha \in \Lambda} (A_\alpha - \alpha) \).

Note that the blocks are of the form

\[
\left\{ (\omega^{i-1+2(j-1)s} + \gamma \omega^{i-1+2(i-1)s})_i : j \in [3] \right\}
\]

for \( (t, i) \in [s] \times [3] \). Suppose otherwise that a point appears twice. That is, there exist \( j, j' \in [3], (t, i), (t', i) \in [s] \times [3] \) with \( t > t' \) such that

\[
\omega^{i-1+2(j-1)s} + \gamma \omega^{i-1+2(i-1)s} = \omega^{i-1+2(j'-1)s} + \gamma \omega^{i-1+2(i-1)s}.
\]

Hence,

\[
\gamma = \frac{\omega^{2(j'-i)s} - \omega^{2(j-i)s+(t-t')}}{\omega^{t-t'} - 1}.
\]

Since \( t \neq t' \), we have \( t - t' \in [s - 1] \). If \( j \neq j' \), this contradicts Condition (B). Otherwise \( j = j' \) implies \( \gamma = -\omega^{2(j-i)s} \) contradicting (A).

Next, observe that \( A_0 = \{(0, i) : i \in [3]\} \). By Claim 6.1, to establish condition (vi) of Definition 6.1, it suffices to show that \( 0 \notin A_\alpha - \alpha \) for \( \alpha \in \Lambda \) and \( i \in [3] \). Suppose otherwise. Then there exists \( (t, j) \in [s] \times [3] \) and \( i \in [3] \) such that

\[
(\omega^{(j-1)s} + \gamma)\omega^{(i-1)s} = 0,
\]

contradicting (A).

Finally, we exhibit that \( S \) is 3-\$\ast\$-colorable with property \( \Pi \) by assigning the block \( A_0 \) color \( \spadesuit \), the blocks \( A_\alpha - \alpha \) for \( \alpha \notin \Lambda \) and \( B_t \) for \( t \in T \) color \( \heartsuit \), and the blocks \( A_\alpha - \alpha \) for \( \alpha \in \Lambda \) color \( \diamondsuit \). Then this assignment satisfies condition (vii) of Definition 6.1. In addition, \( 0_1 \) is a witness for both \( \heartsuit \) and \( \diamondsuit \) and \( \alpha_1 \) is a witness for \( \spadesuit \) for some \( \alpha \neq 0 \), satisfying condition (viii) of Definition 6.1.

**Corollary 6.4.** Let \( q \equiv 1 \mod 6 \). Then a 3-\$\ast\$-colorable GBTD_{1}(3, m) with property \( \Pi \) exists.

**Proof.** This follows from Propositions 6.2 and 6.3.

**Corollary 6.5.** A special GBTD_{1}(3, m) exists for \( m \in \{1, 17, 29, 35, 47, 53, 55\} \), a 3-\$\ast\$-colorable special GBTD_{1}(3, m) with property \( \Pi \) for \( m \in \{9, 11, 23\} \) and a 3-\$\ast\$-colorable RBIBD(15, 3, 1) with property \( \Pi \).

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Proof. A special GBTD$_1(3,1)$ exists trivially. In addition, a 3-$*$-colorable special GBTD$_1(3,9)$ with property $\Pi$ is given by Example 4.4, and a 3-$*$-colorable RBIBD$(15, 3, 1)$ with property $\Pi$ is given by Example 4.1.

For $m \in \{11, 17, 23, 29, 35, 47, 53, 55\}$, apply Proposition 6.2 with special $(\mathbb{Z}_m \times [3])$-GBTD-starters and 3-$*$-colorable special $(\mathbb{Z}_m \times [3])$-GBTD-starters with property $\Pi$ given in [5]. □

6.2. Direct Constructions for an IGBTP$_1([2, 3^*], 2m + w, (m + (w - 1)/2) \times (2m + w - 4); w, (w - 1)/2 \times (w - 4))$

As with GBTDs, we use a set of starters to construct GBTPs. To construct these starters, we need the notion of infinite elements and intransitive starters.

Given an abelian group $\Gamma$, we augment the point set with infinite elements, denoted by $\infty$, where $i$ belongs to some index set $I$. The infinite elements are fixed under addition by elements in $\Gamma$. That is, $\infty_i + \gamma = \infty_j$ for $\gamma \in \Gamma$. Let $w$ be a positive integer and $W_w = \{\infty_j : i \in [w]\}$. So, given a block $A \subset \Gamma \cup W_w$ and $\gamma \in \Gamma$, $A + \gamma = \{a + \gamma : a \in A \setminus W_w\} \cup (A \cap W_w)$.

We also extend the definition of difference lists. For a set system $(\Gamma \cup W_w, S)$, then the difference list of $S$ is given by the multiset

$$\Delta S = \{x - y : x, y \in A \setminus W_w, x \neq y, A \in S\}.$$  

Definition 6.6. Let $m$ be an odd integer with $m \geq 11$. Let $(\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w, S)$ be a $[2, 3]$-uniform set system of size $w - 3 + m$, where

$$S = \{A_i : i \in [(w - 5)/2]\} \cup \{B_i : i \in [(w - 1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\}$$

satisfying $|A_i| = 2$ for $i \in [(w - 5)/2]$, $|B_i| = 2$ for $i \in [(w - 1)/2]$, $|C_0| = 3$, and $|C_i| = 2$ for $i \in \mathbb{Z}_m \setminus \{0\}$.

$S$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter if the following conditions hold:

(i) $\Delta S = \mathbb{Z}_m \times \mathbb{Z}_2 \setminus \{0_0, 0_1\}$,

(ii) $\{j : a_j \in A_i\} = \mathbb{Z}_2$ for $i \in [(w - 5)/2]$,

(iii) $\{B_i : i \in [(w - 1)/2]\} \cup \{C_j : j \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$,

(iv) $|C_i \cap W_w| \leq 1$ for $i \in \mathbb{Z}_m$,

(v) each element in $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$ appears either once or twice in the multiset

$$R = \{0_0, 0_1\} \cup \left( \bigcup_{i \in [(w - 5)/2]} A_i + 0_j \right) \cup \left( \bigcup_{i_j \in \mathbb{Z}_m \times \mathbb{Z}_2} C_i - i_j \right).$$

Proposition 6.7. Suppose there exists a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter. Then there exists an IGBTP$_1([2, 3^*], 2m + w, (m + (w - 1)/2) \times (2m + w - 4); w, (w - 1)/2 \times (w - 4))$.  

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\[X = \mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w,\]
\[A = \{S + j : S \in S \text{ and } j \in \mathbb{Z}_m \times \mathbb{Z}_2 \} \cup \{i_0, i_1 : i \in \mathbb{Z}_m \}.\]

Then \((X, W_w, A)\) is an IRP\((2m + w, K, 1; w)\), whose blocks can be arranged in an \((m + (w - 1)/2) \times (2m + w - 4)\) array as in Fig. 7. We index the rows by \([((w - 1)/2] \cup \mathbb{Z}_m\) and the columns by \([w - 4] \cup (\mathbb{Z}_m \times \mathbb{Z}_2)\).

First, check that the cell \((r, c)\) is empty for \((r, c) \in \[(w - 1)/2] \times [w - 4]\).

For \(j \in \{w - 4\}\), the set of blocks occupying column \(j\) is \(\mathbb{Z}_m \times \mathbb{Z}_2\) by condition (ii) of Definition 6.6. For \(j \in \mathbb{Z}_m \times \mathbb{Z}_2\), first observe that the set of the blocks occupying the column \(0_0\) by condition (iii) of Definition 6.6 is \((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w\). Since the blocks of column \(j\) are translates (by \(j\)) of the blocks in column \(0_0\), the union of the blocks in column \(j\) is also \((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w\).

For \(i \in \[(w - 1)/2\]\), each element in \(\mathbb{Z}_m \times \mathbb{Z}_2\) appears exactly twice in row \(i\) by construction. For \(i \in \mathbb{Z}_m\), let \(R_i\) denote the multiset containing all the points appearing in the blocks of row \(i\). Then \(R_i = R_0 = R_0 + i_0\), for all \(i \in \mathbb{Z}_m\). Hence, it suffices each element in \(X\) appears either once or twice in \(R\), which follows immediately from conditions (v) in Definition 6.6.

**Definition 6.8.** Let \(m\) be an odd integer with \(m \geq 11\). Let \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, S\) be a \((1, 2, 3)\)-uniform set system of size \(7 + 2m\), where

\[S = \{x_0\} \cup \{y_0\} \cup A \cup \{B_i : i \in \{4\}\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\}\]

satisfying \(|A| = 2, |B_i| = 2\) for \(i \in \{4\}\), \(|C_0| = 3, |C_i| = 2\) for \(i \in \mathbb{Z}_m \setminus \{0\}\), and \(|D_i| = 2\) for \(i \in \mathbb{Z}_m\).

\(S\) is called a \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9\)-IGBTP-starter if the following conditions hold:

(i) \(\Delta S = (\mathbb{Z}_m \times \mathbb{Z}_4) \setminus \{0_0, 0_1, 0_2, 0_3\}\),
(ii) \(\{i : a_i \in A\} = \{0, 2\}\),
(iii) \(\{B_i : i \in ((w - 1)/2)\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9\),
(iv) \(|C_i \cap W_9| \leq 1\) and \(|D_i \cap W_9| \leq 1\) for \(i \in \mathbb{Z}_m\),
(v) each element in \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9\) appears either once or twice in the multisets

\[R_0 = \{0_0, 0_1, x_0, x_2, y_0, y_3\} \cup A \cup A + 0_2 \cup \left( \bigcup_{i \in \mathbb{Z}_m, j \in \{0, 2\}} C_i - i_j \right) \cup \left( \bigcup_{i \in \mathbb{Z}_m, j \in \{1, 3\}} D_i - i_j \right),\]
\[R_* = \{0_2, 0_3, x_1, x_3, y_1, y_2\} \cup A \cup A \cup 0_3 \cup \left( \bigcup_{i \in \mathbb{Z}_m, j \in \{1, 3\}} C_i - i_j \right) \cup \left( \bigcup_{i \in \mathbb{Z}_m, j \in \{0, 2\}} D_i - i_j \right).\]

**Proposition 6.9.** Suppose there exists a \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9\)-IGBTP-starter. Then there exists an IGBTP\(_4\)((\{2, 3^*\}, 4m + 9, (2m + 4) \times (4m + 5); 9, 4 \times 5).
Proof. Let

\[ X = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, \]
\[ \mathcal{A} = \{ s + j : s \in S, |s| \neq 1, j \in \mathbb{Z}_m \times \mathbb{Z}_2 \} \cup \{ i_0, i_1 \} \cup \{ i_2, i_3 : i \in \mathbb{Z}_m \} \]
\[ \cup \{ (x + i)_0, (x + i)_2 : i \in \mathbb{Z}_m \} \cup \{ (x + i)_1, (x + i)_3 : i \in \mathbb{Z}_m \} \]
\[ \cup \{ (y + i)_0, (y + i)_2 : i \in \mathbb{Z}_m \} \cup \{ (y + i)_1, (y + i)_3 : i \in \mathbb{Z}_m \}. \]

Then \( (X, W_9, \mathcal{A}) \) is an IRP\((4m + 9, K, 1; 9)\), whose blocks can be arranged in a \((2m + 4) \times (4m + 5)\) array as in Fig. 6. We index the rows by \([4] \cup (\mathbb{Z}_m \times \{ \circ, \bullet \})\) and the columns by \([5] \cup (\mathbb{Z}_m \times \mathbb{Z}_4)\).

First, check that the cell \((r, c)\) is empty for \((r, c) \in [4] \times [5].\)

For \(j \in [5],\) the set of blocks occupying column \(j\) is \(\mathbb{Z}_m \times \mathbb{Z}_4\) by condition (ii) of Definition 6.8. For \(j \in \mathbb{Z}_m \times \mathbb{Z}_4,\) first observe that the set of the blocks occupying the column \(0_0\) by condition (iii) of Definition 6.8 is \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9.\) Since the blocks of column \(j\) are translates (by \(j\)) of the blocks in column \(0_0,\) the union of the blocks in column \(j\) is also \((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9.\)

For \(i \in [4],\) each element in \(\mathbb{Z}_m \times \mathbb{Z}_4\) appears exactly twice in row \(i\) by construction. For \((i, \ast) \in \mathbb{Z}_m \times \{ \circ, \bullet \},\) let \(R_{(i, \ast)}\) denote the multiset containing all the points appearing in the blocks of row \((i, \ast).\) Then \(R_{(0, \ast)} = R_0\) and \(R_{(i, \ast)} = R_{(0, \ast)} + i_0,\) for all \(i \in \mathbb{Z}_m.\) Hence, it suffices each element in \(X\) appears either once or twice in \(R_4,\) which follows immediately from conditions (v) in Definition 6.8.

\[ \square \]

Corollary 6.10. An IGBTP\(_1([2, 3^*], 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)\) exists for \(m \in \{ s : 10 \leq s \leq 45 \} \cup \{ 47, 49, 53, 57, 77 \} \setminus \{ 16, 20, 24, 28, 36, 40, 44 \},\) and an IGBTP\(_1([2, 3^*], 2m + 11, (m + 5) \times (2m + 7); 11, 5 \times 7)\) exists for \(m \in \{ 15, 19, 23, 27, 31, 35, 45, 49 \}.\)

Proof. The required \(((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_9)\)-IGBTP-starter for \(m \in \{ s : 11 \leq s \leq 49, s \text{ odd} \} \cup \{ 53, 57, 77 \}\) and \(((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)\)-IGBTP-starter for \(m \in \{ s : 5 \leq s \leq 21, s \text{ odd} \}\) is given in [5] and we apply Propositions 6.7 and 6.9 to obtain the corresponding IGBTP.

Similarly, to construct an IGBTP\(_1([2, 3^*], 2m + 11, (m + 5) \times (2m + 7); 11, 5 \times 7)\) for \(m \in \{ 15, 19, 23, 27, 31, 35, 45, 49 \},\) we apply Proposition 6.7 to \((\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_1)\)-IGBTP-starters listed in [5].

It remains to construct an IGBTP\(_1([2, 3^*], 33, 16 \times 29; 9, 4 \times 5)\). Consider \(((\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W_9, S),\) a \([2, 3]\)-uniform set system of size 36, where \(S\) comprise the blocks below:

\[ A_1 = \{ 1_0, 1_2 \} \quad A_2 = \{ 1_1, 1_5 \} \quad A_3 = \{ 0_0, 0_4 \} \quad A_4 = \{ 1_3, 1_6 \} \]
\[ A_5 = \{ 0_3, 0_5 \} \quad A_6 = \{ 1_1, 1_3 \} \quad A_7 = \{ 1_4, 1_7 \} \quad A_8 = \{ 0_1, 0_6 \} \]
\[ A_9 = \{ 0_0, 0_5 \} \quad A_{10} = \{ 0_2, 0_4 \} \quad A_{11} = \{ 1_4, 1_6 \} \quad A_{12} = \{ 1_0, 1_3 \} \]
\[ A_{13} = \{ 0_2, 0_5 \} \quad A_{14} = \{ 1_2, 1_7 \} \quad A_{15} = \{ 0_1, 0_7 \} \quad A_{16} = \{ 1_5, 1_7 \} \]
\[ A_{17} = \{ 0_2, 0_6 \} \quad A_{18} = \{ 0_3, 0_7 \} \quad A_{19} = \{ 1_1, 1_4 \} \quad A_{20} = \{ 1_0, 1_6 \} \]
\[ B_1 = \{ 0_0, 0_1 \} \quad B_2 = \{ 0_3, 1_5 \} \quad B_3 = \{ 1_1, 2_4 \} \quad B_4 = \{ 0_7, 1_3 \} \]
\[ C_1 = \{ 1_0, 2_0, 2_6 \} \quad C_1 = \{ 1_0, 2_1 \} \quad C_2 = \{ 1_0, 2_1 \} \quad C_3 = \{ 0_2, 0_8 \} \]
\[ C_4 = \{ 0_2, 0_8 \} \quad C_7 = \{ 0_4, 0_8 \} \quad C_2 = \{ 1_2, 1_8 \} \quad C_3 = \{ 1_2, 1_8 \} \]
\[ C_0 = \{ 2_0, 2_4 \} \quad C_1 = \{ 2_3, 2_8 \} \quad C_2 = \{ 1_6, 1_8 \} \quad C_3 = \{ 1_6, 1_8 \} \]
\[ C_4 = \{ 2_7, 2_9 \} \quad C_4 = \{ 2_7, 2_9 \} \quad C_4 = \{ 2_7, 2_9 \} \quad C_4 = \{ 2_7, 2_9 \}. \]
Let
\[ X = (\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W \]
\[ \mathcal{A} = \{ S + j : S \in \mathcal{S}, j \in \mathbb{Z}_3 \times \mathbb{Z}_8 \}. \]

Then \((X, W, \mathcal{A})\) is an IRP(33, \{2, 3\}, 1; 9), whose blocks can be arranged in a 16 × 29 array as in Fig. 8. It can be readily verified that this arrangement results in an IGBTP\(_1\)(\{(2, 3\}, 33, 16 × 29; 9, 4 × 5). □

### 6.3. Direct Constructions for FrGBTDs

**Lemma 6.11.** There exists an FrGBTD\((2, 2^t)\) for \(t \in \{4, 5\} \).

**Proof.** The desired FrGBTDs are given in Figs. 9 and 10. □

**Definition 6.12.** Let \(t\) be a positive integer, and let \(I = [t - 1] \times [2]\). Let \((\mathbb{Z}_{3t} \times [2], \mathcal{S})\) be a 3-uniform set system of size \(2(t - 1)\), where \(\mathcal{S} = \{A_i : i \in I\}\). \(\mathcal{S}\) is called a \((\mathbb{Z}_{3t} \times [2])\)-FrGBTD-starter if the following conditions hold:

1. \(\Delta_{ij} \mathcal{S} = \mathbb{Z}_{3t} \setminus \{0, t, 2t\}\) for \(i, j \in [2]\),
2. \(\bigcup_{i \in I} A_i = (\mathbb{Z}_{3t} \setminus \{0, t, 2t\}) \times [2]\),
3. for \(j \in [2]\), each element in \((\mathbb{Z}_t \setminus \{0\}) \times [2]\) appears either once or twice in the multiset
   \[ R_j = \bigcup_{i=1}^{t-1} A_{(i,j)} - i \mod t, \]

4. \(r \in (\mathbb{Z}_t \setminus \{0\}) \times [2]\) for each \(r \in R_1 \cup R_2\).

**Proposition 6.13.** If a \((\mathbb{Z}_{3t} \times [2], 6^t)\)-FrGBTD-starter exists, then an FrGBTD\((3, 6^t)\) exists.

**Proof.** Let
\[ X = \mathbb{Z}_{3t} \times [2], \]
\[ \mathcal{G} = \{ G_i = \{ i_1, (t + i)_1, (2t + i)_1, i_2, (t + i)_2, (2t + i)_2 : i \in \mathbb{Z}_t \}, \]
\[ \mathcal{A} = \{ A_i + j : i \in I \text{ and } j \in \mathbb{Z}_{3t} \}. \]

Then \((X, \mathcal{G}, \mathcal{A})\) is a \(\{3\}\)-GDD of type \(6^t\), whose blocks can be arranged in a \(2t \times 3t\) array, with rows and columns indexed by \(\mathbb{Z}_t \times [2]\) and \(\mathbb{Z}_{3t}\), respectively, as follows: the block \(A_{(i,j)} + k\) is placed in cell \(((i + k, j), k)\).

The set of blocks occupying column zero are \(\{ A_i : i \in I \}\) and by condition (ii) of Definition 6.12, \(\bigcup_{i \in I} A_i = X \setminus G_0\). For other \(j \in \mathbb{Z}_{3t}\), observe that the blocks occupying column \(j\) are translates (by \(j\)) of the blocks in column zero, and hence the union of the columns in column \(j\) is \(X \setminus G_j\), where \(j' \equiv j \mod t\).

For \((i, j) \in \mathbb{Z}_t \times [2]\), let \(R_{(i,j)}\) denote the multiset containing all the points appearing in the blocks of row \((i, j)\). Then \(R_{(0,j)} = R_{(0,j)} + i\), for all \(i \in \mathbb{Z}_t\). Hence, it suffices to check that each element of \(X \setminus G_0\) appears either once or twice in \(R_{(0,j)}\) and the elements of \(R_{(0,j)}\) belong to \(X \setminus G_j\) for \(j \in [2]\). This, however, follows immediately from conditions (iii) and (iv) in Definition 6.12, since \(R_{(0,j)} = R_j \cup (R_j + t) \cup (R_j + 2t)\) for \(j \in [2]\). □
where $W$ is a $4 \times 5$ empty array, $A$ is a $12 \times 5$ array,

$$
\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 \\
A_1 + 1 & A_2 + 1 & A_3 + 1 & A_4 + 1 & A_5 + 1 \\
A_1 + 2 & A_2 + 2 & A_3 + 2 & A_4 + 2 & A_5 + 2 \\
A_6 & A_7 & A_8 & A_9 & A_{10} \\
A_6 + 1 & A_7 + 1 & A_8 + 1 & A_9 + 1 & A_{10} + 1 \\
A_6 + 2 & A_7 + 2 & A_8 + 2 & A_9 + 2 & A_{10} + 2 \\
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{11} + 1 & A_{12} + 1 & A_{13} + 1 & A_{14} + 1 & A_{15} + 1 \\
A_{11} + 2 & A_{12} + 2 & A_{13} + 2 & A_{14} + 2 & A_{15} + 2 \\
A_{16} & A_{17} & A_{18} & A_{19} & A_{20} \\
A_{16} + 1 & A_{17} + 1 & A_{18} + 1 & A_{19} + 1 & A_{20} + 1 \\
A_{16} + 2 & A_{17} + 2 & A_{18} + 2 & A_{19} + 2 & A_{20} + 2
\end{array}
$$

$B$ is a $4 \times 3$ array,

$$
\begin{array}{ccc}
B_1 & B_1 + 1 & B_1 + 2 \\
B_2 & B_2 + 1 & B_2 + 2 \\
B_3 & B_3 + 1 & B_3 + 2 \\
B_4 & B_4 + 1 & B_4 + 2
\end{array}
$$

$C_i$ for $i \in [4]$ is a $3 \times 3$ array,

$$
\begin{array}{c}
C_0^i & C_2^i + 1 & C_1^i + 2 \\
C_1^i & C_0^i + 1 & C_2^i + 2 \\
C_2^i & C_1^i + 1 & C_0^i + 2
\end{array}
$$

**FIGURE 8.** An IGBTP$_4(\{2, 3^+\}, 33, 16 \times 29; 9, 4 \times 5)$.

<table>
<thead>
<tr>
<th>2.3</th>
<th>6.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.7</td>
<td>1.3</td>
</tr>
<tr>
<td>1.6</td>
<td>5.2</td>
</tr>
</tbody>
</table>

**FIGURE 9.** An FrGBTD$_4(2, 2^4) (X, G, A)$, where $X = \mathbb{Z}_8$ and $G = \{i, 4 + i : i \in \mathbb{Z}_4\}$.

*Journal of Combinatorial Designs* DOI 10.1002/jcd
\[ \begin{array}{ccccccccc}
7,4 & 2,9 & 7,9 & 2,4 & 3,4 & 8,9 & 6,2 & 1,7 & 1,8 & 6,3 \\
3,9 & 8,4 & 8,5 & 3,0 & 9,1 & 4,6 & 5,6 & 0,1 & 7,3 & 2,8 \\
1,2 & 6,7 & 4,0 & 9,5 & 9,6 & 4,1 & 5,2 & 0,2 & 5,7 & 2,9 \\
6,8 & 1,3 & 2,3 & 7,8 & 5,1 & 0,6 & 0,7 & 5,2 & 3,0 & 8,0 \\
\end{array} \]

**FIGURE 10.** An FrGBTD\((2, 2^5)\) \((X, G, A)\), where \(X = \mathbb{Z}_{10}\) and \(G = \{i, 5 + i : i \in \mathbb{Z}_5\}\).

**Corollary 6.14.** There exist an FrGBTD\((3, 6^t)\) for all \(t \in \{5, 6, 7, 8\}\), an FrGBTD\((3, 24^t)\) for all \(t \in \{5, 8\}\) and an FrGBTD\((3, 30^t)\) for all \(t \in \{5, 7\}\).

**Proof.** An FrGBTD\((3, 6^6)\) is given by Example 4.5. An FrGBTD\((3, 6^t)\) for \(t \in \{5, 7\}\) exists by applying Proposition 6.13 with FrGBTD-starters given in [5].

The existence of an FrGBTD\((3, 24^t)\), \(t \in \{5, 8\}\) follows by applying Proposition 5.7 with an FrGBTD\((3, 6^t)\) (constructed in this proof) and a DRTD\((3, 4)\), whose existence is provided by Corollary 4.11. The existence of an FrGBTD\((3, 30^t)\), \(t \in \{5, 7\}\) follows by applying Proposition 5.7 similarly.

To prove the existence of an FrGBTD\((3, 6^8)\), consider \((\mathbb{Z}_{48}, S)\), a \(\{3\}\)-uniform set system of size 7, where \(S\) comprise the blocks below:

\[ A_1 = \{2, 3, 5\} \quad A_2 = \{4, 14, 31\} \quad A_3 = \{9, 22, 45\} \quad A_4 = \{15, 34, 43\} \]
\[ A_5 = \{20, 35, 42\} \quad A_6 = \{13, 17, 47\} \quad A_7 = \{1, 6, 12\}. \]

Observe that \(S\) satisfies the following conditions:

(i) \(\Delta S = \mathbb{Z}_{48} \setminus \{0, 8, 16, 24, 32, 40\}\),
(ii) \(\bigcup_{i \in [7]} A_i \mod 24 = \mathbb{Z}_{24} \setminus \{0, 8, 16\}\),
(iii) each element in \(\mathbb{Z}_{16} \setminus \{0, 8\}\) appears either once or twice in the multiset
\[ R = \bigcup_{i \in [7]} A_i - i \mod 16, \]
(iv) \(r \in \mathbb{Z}_{16} \setminus \{0, 8\}\) for each \(r \in R\).

Further, let

\[ X = \mathbb{Z}_{48}, \]
\[ G = \{i + 8k : k \in \mathbb{Z}_6 : i \in \mathbb{Z}_8\}, \]
\[ A = \{A_i + j : i \in [7] \text{ and } j \in \mathbb{Z}_{48}\}. \]

Then \((X, G, A)\) is a \(\{3\}\)-GDD of type \(6^8\), whose blocks can be arranged in a \(16 \times 24\) array, with rows and columns are indexed by \(\mathbb{Z}_{16}\) and \(\mathbb{Z}_{24}\), respectively, as follows: the block \(A_i + j\) is placed in cell \((i + j, j)\). This array can be verified to be an FrGBTD\((3, 6^8)\). \(\square\)

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TABLE II. Existence of special GBTD$_1(3, m)$.

<table>
<thead>
<tr>
<th>Authority</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary 6.5</td>
<td>9, 11, 17, 23, 29, 35, 47, 53, 55</td>
</tr>
<tr>
<td>Lemma 7.1</td>
<td>7, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49, 57, 61, 63, 67, 69, 73, 75</td>
</tr>
<tr>
<td>Corollary 5.6 with $(g, t)$ in</td>
<td>41, 51, 65, 71</td>
</tr>
<tr>
<td>${8, 5}, (5, 10), (8, 8), (7, 10)}$</td>
<td></td>
</tr>
<tr>
<td>Lemma 7.2 with $n = 5, g_1 = 4$</td>
<td>59</td>
</tr>
<tr>
<td>Lemma 7.2 with $n = 7, g_1, g_2 \in {0} \cup {t : 3 \leq t \leq 7}$</td>
<td>{s : 77 \leq s \leq 95, s odd}</td>
</tr>
</tbody>
</table>

7. EXISTENCE OF GBTDS AND GBTPs

We apply the recursive constructions in Section 5 using the small designs constructed directly in Section 6 to completely settle the existence of GBTD$_1(3, m)$ and GBTP$_1([2, 3^*]; 2m + 1, m \times (2m - 3))$.

7.1. Existence of GBTD$_1(3, m)$

**Lemma 7.1.** There exists a special GBTD$_1(3, 3')q$ for all $r \geq 0$ and $q \in Q$, where $Q = \{q : q \equiv 1 \mod 6$ is a prime power$\} \cup \{5, 9, 11, 23\}$, except when $(r, q) = (0, 5)$.

*Proof.* Existence of a special GBTD$_1(3, q)$ for all $q \in Q \setminus \{5\}$ is provided by Corollaries 6.4 and 6.5. These GBTDs are all 3-$*$colorable with property $\Pi$. The lemma then follows by considering these GBTDs as RBIBDs and applying Corollary 5.2. \qed

**Lemma 7.2.** Let $s \in [2]$ and suppose there exists a TD$(5 + s, n)$. If $0 \leq g_1 \leq n, i \in [s]$ and that there exists a special GBTD$_1(3, m)$ for all $m \in \{2n + 1\} \cup \{2g_i + 1 : i \in [s]\}$, then there exists a special GBTD$_1(3, 10n + 1 + 2 \sum_{i=1}^{s} g_i)$.

*Proof.* By Corollary 6.14, there exists an FrGBTD$(3, 6')$ for all $t \in \{5, 6, 7\}$. By Proposition 5.9, there exists an FrGBTD$(3, (6n)^5(6g_1) \cdots (6g_s))$. Now apply Corollary 5.5 to obtain a special GBTD$_1(3, 10n + 1 + 2 \sum_{i=1}^{s} g_i)$. \qed

**Lemma 7.3.** A special GBTD$_1(3, m)$ exists for odd $m \geq 7$.

*Proof.* First, a special GBTD$_1(3, m)$ can be constructed for odd $m, 7 \leq m \leq 95$. Details are provided in Table II.

We then prove the lemma by induction on $m \geq 97$.

Let $E = \{t : t \geq 9\} \setminus \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$. By Theorem 4.8, a TD$(7, n)$ exists for any $n \in E$. If there exists a special GBTD$_1(3, m')$ for odd $m', 7 \leq m' \leq 2n + 1$, then apply Lemma 7.2 with $3 \leq g_1, g_2 \leq n$ to obtain a special GBTD$_1(3, m)$ for odd $m, 10n + 7 \leq m \leq 14n + 1$.

Hence, take $n = 9$ to obtain a special GBTD$_1(3, 97)$.

Suppose there exists a GBTD$_1(3, m')$ for all odd $m' < m$. Then there exists $n \in E$ with $10n + 7 \leq m \leq 14n + 1$. Suppose otherwise. Then there exists $n_1 \in E$ such that $14n_1 + 1 < 10n_2 + 7$ for all $n_2 > n_1$ and $n_2 \in E$. This, together with the fact that $n_1 \geq 9$,
TABLE III. Existence of IGTP(1(2, 3*); 2m + 9, (m + 4) × (2m + 5); 4 × 5).

<table>
<thead>
<tr>
<th>Authority</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary 6.10</td>
<td>{s : 10 ≤ s ≤ 57 } \setminus {16, 20, 24, 28, 32, 36, 40, 44, 48, 50, 52, 54, 55, 56}</td>
</tr>
<tr>
<td>Lemma 7.4 with ((n, g) \in {(10, 0), (11, 0), (12, 0), (13, 0), (11, 10), (11, 11), (14, 0)})</td>
<td>40, 44, 48, 52, 54, 55, 56</td>
</tr>
</tbody>
</table>

implies that \(n_2 - n_1 > 3\) for all \(n_2 \in E\) and \(n_2 > n_1\). However, a quick check on \(E\) gives a contradiction.

Since \(n \in E\) and there exists a special GBTD(3, \(m'\)) for all \(m' \leq 2n + 1 < 10n + 7 \leq m\) (induction hypothesis), there exists a special GBTD(3, \(m\)) and induction is complete. \(\square\)

Lemma 7.3 shows that a GBTD(3, \(m\)) exists for all odd \(m \neq 3, 5\). Theorem 2.3 (vi) now follows.

7.2. Existence of GBTP(1(2, 3*); 2m + 1, m × (2m - 3))

**Lemma 7.4.** Suppose there exists a TD(5, \(n\)). Suppose \(0 ≤ g ≤ n\) and that there exists an IGTP(1(2, 3*), 2m + 9, (m + 4) × (2m + 5); 9, 4 × 5) for \(m \in \{n, g\}\). Then there exists an IGTP(1(2, 3*), 2M + 9, (M + 4) × (2M + 5); 9, 4 × 5), where \(M = 4n + g\).

**Proof.** By Lemma 6.11, there exists an FrGBTD(2, 2'), for all \(t \in \{4, 5\}\). By Proposition 5.9, there exists an FrGBTD(2, (2n)\((2g)\)) . Now apply Proposition 5.4 to obtain an IGTP(1(2, 3*), 2M + 9, (M + 4) × (2M + 5); 9, 4 × 5). \(\square\)

**Lemma 7.5.** There exists an IGTP(1(2, 3*), 2m + 9, (m + 4) × (2m + 5); 9, 4 × 5) for any \(m \geq 10\), except possibly for \(m \in \{16, 20, 24, 28, 32, 36, 46, 50\}\).

**Proof.** Let \(E = \{16, 20, 24, 28, 32, 36, 46, 50\}\). An IGTP(1(2, 3*), 2m + 9, (m + 4) × (2m + 5); 9, 4 × 5) can be constructed for \(10 ≤ m ≤ 57\) and \(m \not\in E \cup \{51\}\). Details are provided in Table III. When \(m = 51\), consider a TD(5, 11) and delete four points from a block to form a \(4, 5\)-GDD of type \(10^411\). Proposition 5.8 yields an FrGBTD(2, 20\(12\)) and hence, Proposition 5.4 yields an IGTP(1(2, 3*), 2m + 9, (m + 4) × (2m + 5); 9, 4 × 5) with \(m = 51\).

We then prove the lemma by induction on \(m ≥ 57\). Let \(E' = \{4n + g : n \in E, 10 ≤ g ≤ 13\}\) and assume the lemma is true for \(n < m\).

When \(m \not\in E'\), then write \(m = 4n + g\) with \(13 ≤ n < m\), \(n \not\in E\) and \(g \in \{10, 11, 12, 13\}\). Since a TD(5, \(n\)) that exists by Theorem 4.8, applying Lemma 7.4 with the corresponding \(n\) and \(g\), we obtain the desired IGTP.

When \(m \in E'\), we have two cases.

- If \(m = 77\), the required IGTP is given by Corollary 6.10.
- Otherwise, apply Lemma 7.4 with \((n, g)\) taking values in \(\{(15, 14), (15, 15), (19, 0), (18, 18), (19, 15), (23, 0), (19, 17), (22, 18), (22, 19), (27, 0), (22, 21), (25, 22), (25, 50)\}\).
This completes the induction. □

Lemma 7.6. A GBTP\(_1(\{2, 3^*\}, 2m + 1, m \times (2m - 3))\) exists for \(m \geq 4\), except possibly for \(m \in \{12, 13\}\).

Proof. A GBTP\(_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))\) can be found via computer search for \(4 \leq m \leq 11\). The GBTPs are listed in [5].

For \(m \in \{20, 24, 28, 32, 36, 40, 50, 54\}\), set \(M = m - 5\) and we apply Proposition 5.3 with the GBTP\(_1(\{2, 3^*\}, 11, 5 \times 7)\) and the IGBTP\(_1(\{2, 3^*\}, 2M + 11, (M + 5) \times (2M + 7); 11, 5 \times 7)\) constructed in Corollary 6.10.

Finally, for \(m \geq 14\) and \(m \notin \{20, 24, 28, 32, 36, 40, 50, 54\}\), set \(M = m - 4\) and apply Proposition 5.3 with GBTP\(_1(\{2, 3^*\}, 9, 4 \times 5)\) and the IGBTP\(_1(\{2, 3^*\}, 2M + 9, (M + 4) \times (2M + 5); 9, 4 \times 5)\) constructed in Lemma 7.5. □

Lemma 7.6 shows that a GBTP\(_1(\{2, 3^*\}, 2m + 1, m \times (2m - 3))\) exists for all \(m \geq 4\), except possibly for \(m \in \{12, 13\}\). Theorem 2.3 (vii) now follows.

8. CONCLUSION

In this paper, we establish infinite families of ESWCs, whose code lengths are greater than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as length grows. The construction method used is combinatorial and reveals interesting interplays with equivalent combinatorial designs called GBTPs. These have enabled us to borrow ideas from combinatorial design theory to construct ESWCs. In return, questions on ESWCs offer new problems to combinatorial design theory. We expect this symbiosis to deepen.

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