The Effects of Adjustment Costs and Uncertainty on Investment Dynamics and Capital Accumulation*

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Abstract

This paper provides a unified framework to study how capital adjustment costs and uncertainty affect investment dynamics and capital accumulation. It considers an ongoing firm with stochastic downward sloping demand curve and facing three possible forms of adjustment costs: complete or partial irreversibility, fixed costs of undertaking any investment and the traditional quadratic adjustment costs. The quantities of interest are the impact effects of demand shocks on capital adjustment in the short run, and the expected capital stock level in the long run, under different forms of adjustment costs, and at different levels of uncertainty.

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1 Introduction

According to the static neoclassical producer theory, a firm’s optimal investment is to equalize the marginal revenue product of capital ($MPK$, hereafter) to the user cost of capital ($ucc$, hereafter), as derived by Jorgenson (1963). Two important features of investment turn this static problem into dynamic: uncertainty about future economic environment and costly adjustment of capital stock. Without uncertainty, a firm can follow a deterministic optimal investment path even in the presence of adjustment costs. Without adjustment costs, a firm can instantaneously and costlessly adjust capital stock in each period even in the presence of uncertainty. Therefore studying the effects of uncertainty and adjustment costs on firm’s investment decision and capital accumulation is crucial in understanding investment.

This paper provides a unified framework to study these short-run and long-run effects on an ongoing firm. It considers a firm facing stochastic downward sloping demand curve\(^1\) and three possible forms of adjustment costs—complete or partial irreversibility, fixed costs of undertaking any investment, and the traditional quadratic adjustment costs\(^2\). By construction, in the absence of any adjustment costs, both investment dynamics and capital accumulation are invariant to the level of uncertainty in such a framework. This provides a useful benchmark to investigate two sequential questions: first, what are the effects of different forms of adjustment costs, compared with the frictionless benchmark; and second, what are the effects of uncertainty through these adjustment costs?

In the past three decades the investment literature has paid much attention to irreversibility and achieved important insights. When investment is irreversible, the optimal investment policy is to purchase capital only as needed to prevent the $MPK$ from rising above an optimally derived hurdle. The hurdle, which is the $ucc$ appropriately defined to take account of irreversibility and uncertainty, is higher than the Jorgensonian $ucc$ and hence predicts less investment. This result is known as the ‘real option’ effect in the option approach (Bertola, 1988; Pindyck, 1988; Dixit and Pindyck, 1994); or the ‘user cost’ effect in the $q$-approach (Abel and Eberly, 1996); and is unified in Abel, Dixit, Eberly and Pindyck (1996). A related result is that an increase in uncertainty facing the firm tends to increase the $ucc$ under irreversibility, which further reduces the optimal investment. This relationship is formalized as a negative effect of uncertainty on the impact effect of positive demand shocks on investment dynamics in Bond, Bloom and Van Reenen (2007) and Bloom (2009).

Nevertheless, the negative short-run effects do not necessarily imply lower capital

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\(^1\)As emphasized in Caballero (1991), Pindyck (1993) and Abel and Eberly (1996), in analyzing the effect of uncertainty, it is important for the $MPK$ of the firm to be a decreasing function of the capital stock. Otherwise the future $MPK$ are unaffected by today’s investment, so the link from today’s investment to future returns is broken.

\(^2\)In the early investment literature, such as Hayashi (1982), the adjustment costs merely referred to what is called convex or quadratic adjustment costs more recently. Irreversibility stood as another type of friction that affects firm’s investment decision. Fixed adjustment costs became the focus of new interest since 1990s. Abel and Eberly (1994) include both traditional quadratic adjustment costs and irreversibility, together with a fixed component of capital adjustment costs into an augmented adjustment cost function. To distinguish from quadratic adjustment costs, irreversibility and fixed costs are sometimes called non-convex adjustment costs.
stock in the long run. This is because if the $MPK$ is unusually low at date $t$, the firm would like to sell some of its capital at a positive price. However, under irreversibility, the firm cannot sell capital, and it is constrained by its own past investment behavior to have a capital stock that is higher than it would choose if it could start fresh at date $t$. Abel and Eberly (1999) refer to this effect as the ‘hangover’ effect to indicate the dependence of the current capital stock on past behavior.

In the special case of complete irreversibility, no depreciation, a Brownian motion demand process and an infinite time horizon, Abel and Eberly (1999) demonstrate analytically that the user cost effect and hangover effect have opposite implications for the expected long-run capital stock. Irreversibility may increase or decrease capital accumulation relative to the frictionless benchmark. Furthermore, an increase in uncertainty can either increase or decrease the long-run capital stock under irreversibility relative to that under reversibility.

Recently more and more empirical evidence has highlighted the importance of other forms of adjustment costs. For example, Cooper and Haltiwanger (2006) find the evidence of fixed adjustment costs in plant level data; Eberly, Rebelo and Vincet (2008) emphasize the importance of quadratic adjustment costs in firm level data; and Bond, Söderbom and Wu (2008) find the significance of both fixed and quadratic adjustment costs at the firm level. However, compared with irreversibility, there is little theoretical work about how uncertainty affects firm’s investment dynamics and capital accumulation in the presence of fixed and quadratic adjustment costs. No analytical solution to the investment model with generalized adjustment costs is the main reason for this gap. Lack of well-defined distinct short-run and long-run quantities of interest is another cause of the gap.

Using numerical dynamic programming methods, this paper solves a generalized investment model with complete and partial irreversibility, fixed costs of investment and quadratic adjustment costs. Following Bloom (2009), the impact effect of positive demand shocks on capital adjustment is defined as the quantity of interest for the short-run dynamics. Following Abel and Eberly (1999), the ratio of the expected capital stock level with adjustment costs to the expected capital stock level without adjustment costs is defined as the quantity of interest for the long-run accumulation.

Concerning the short-run effects, the presence of complete irreversibility, partial irreversibility and quadratic adjustment costs all dampens the responsiveness of investment to new information about demand. Furthermore, in the presence of complete irreversibility, partial irreversibility and fixed adjustment costs, the impact effect of positive demand shocks on capital adjustment is a non-increasing function of uncertainty. This confirms the findings in Bloom (2009) for partial irreversibility but also highlights the importance of other forms of non-convex adjustment costs.

Concerning the long-run effects, the numerical solution in this paper replicates the analytical finding in Abel and Eberly (1999) for the special case of complete irreversibility. Similar properties are found for partial irreversibility, in the sense that the presence of partial irreversibility could either increase or decrease the expected long-run capital stock relative to that under frictionless case; and uncertainty does not ease the ambiguity but rather deepens it. In contrast, in the presence of quadratic adjustment costs, the expected long-run capital stock is unambiguously lower than the frictionless level, due to a $ucc$ that
is higher than the Jorgensonian ucc. Furthermore, this user cost effect gets stronger with an increase in uncertainty hence further reduces the expected long-run capital stock at a higher level of uncertainty. The fixed adjustment costs have the same effect as quadratic adjustment costs at complete certainty but the same effect as partial irreversibility in an uncertain environment.

The numerical methods also allow comparative statics about the effects of other model parameters on these findings. In particular, this paper examines the role of the demand growth rate, the discount rate, the capital share in the production function, the demand elasticity, the depreciation rate and the serial correlation parameter in a trend stationary demand process.

The rest of the paper is organized as follows. Section 2 constructs an investment model under uncertainty and characterizes the optimal investment decision in the presence of different forms of adjustment costs. Section 3 investigates the effects of adjustment costs and uncertainty on both short-run capital adjustment and long-run capital accumulation. The effects of other model parameters are presented in Section 4. Section 5 offers concluding remarks.

2 An Investment Model under Uncertainty

This section sets up an investment model for a firm operating under uncertainty. The functional forms are chosen strictly following Abel and Eberly (1999), except for two variations. First, this section assumes a discrete rather than continuous timing in order to solve the model using standard numerical methods. Second, it allows depreciation of capital stock and three forms of capital adjustment costs. This model therefore nests Abel and Eberly (1999) as a special benchmark case but is also general enough to incorporate other cases in this literature.

2.1 Short-run Profit Optimization

Time is discrete and horizon is infinite. By paying capital adjustment costs, new investment $I_t$ contributes to productive capital $\hat{K}_t$ immediately in period $t$, which depreciates at the end of each period.\(^3\)

Assumption 1 Timing:

The law of motion for capital stock is

$$K_{t+1} = (1 - \delta) (K_t + I_t) \equiv (1 - \delta) \hat{K}_t$$

where $\delta$ is the constant depreciation rate.

\(^3\)Compared with alternative lagged timing assumption, such as $K_{t+1} = (1 - \delta)K_t + I_t$, and only $K_t$ is productive in period $t$, Assumption 1 does not affect the qualitative implications of the model, but allows for a closed-form solution to the investment problem in the frictionless case, which does not involve any expectation term. This provides a convenient benchmark for studying the effects of capital adjustment costs. In the special case of continuous timing and no depreciation as assumed in Abel and Eberly (1999), this timing difference vanishes.
Consider a firm that uses capital stock $\tilde{K}_t$ and labor $L_t$ to produce nonstorable output $Q_t$, according to a nonstochastic constant returns to scale Cobb-Douglas technology.

**Assumption 2 Production:**

The production function is

$$Q_t = L_t^{1-\beta} \tilde{K}_t^\beta$$  \hspace{1cm} (2)

where the capital share $\beta$ satisfies $0 < \beta < 1$.

The firm faces an isoelastic, downward-sloping, stochastic demand curve. Denote $X_t$ as the random component in demand, which can be interpreted as changes in the quantity demanded $Q_t$ for any given price of output $P_t$, and is called ‘horizontal demand shocks’ in Abel and Eberly (1999).

**Assumption 3 Demand:**

The demand schedule is

$$Q_t = X_t P_t^{-\varepsilon}$$  \hspace{1cm} (3)

where $-\varepsilon < -1$ is the demand elasticity with respect to price.

The demand shift parameter $X_t$ is the only source of uncertainty in this model. Abel and Eberly (1999) assume $X_t$ evolves exogenously according to a geometric Brownian motion with mean $\mu t$ and variance $\sigma^2 t$. By Ito’s Lemma, this implies the log of $X_t$ follows a Brownian motion with mean $(\mu - 0.5\sigma^2) t$ and variance $\sigma^2 t$. The discrete time analogue of this process is described in the following assumption:

**Assumption 4 Demand Stochastic:**

The law of motion for $X_t$ is

$$x_t \equiv \ln X_t = x_{t-1} + \bar{\mu} + \varepsilon \hspace{1cm} (4)$$

where $\bar{\mu} = \mu - 0.5\sigma^2 > 0$, $\varepsilon_t = \sigma \varepsilon_t$, $\varepsilon_t \sim i.i.d. \ N(0,1)$, and $x_0 = 0$.

Firm making decisions in period $t$ knows $X_t$ and the parameter values of $X_0$, $\mu$ and $\sigma$, but are uncertain about future levels of demand which depend on future realizations of the demand shocks $\varepsilon_t$. The standard deviation of these demand shocks $\sigma$ therefore measures the level of uncertainty faced by the firm. The condition $\mu > 0.5\sigma^2$ guarantees that the $MPK$ has a non-degenerate ergodic distribution, as restricted in the Eq. (7) of Abel and Eberly (1999).

Labor is a variable input hence is adjusted instantaneously and costlessly. In each period, for given capital stock and demand realization, the firm chooses labor $L_t$ to maximize its instantaneous operating profit $P_t Q_t - w L_t$, where $w$ is a constant wage rate.

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4In the investment literature, Hartman (1972), Abel (1983) and Caballero (1991) highlight the effect of uncertainty on the expected investment expenditure through the curvature of $MPK$ to the stochastic variable that characterizes uncertainty. Compared with the alternative ‘vertical demand shocks’, within this class of model, the specification for the horizontal demand shocks effectively isolate this Hartman-Abel-Caballero effect and allow this paper to focus on the effect of uncertainty through the sole channel of capital adjustment costs. Bond, Söderbom and Wu (2008) offer a structural estimation for the effects of uncertainty through both the Hartman-Abel-Caballero effect and capital adjustment costs effect.
Lemma 1 Operating Profit:

The maximized value of operating profit is given by

\[ \pi(X_t, \hat{K}_t) = \frac{h}{1 - \gamma} X_t^\gamma \hat{K}_t^{1-\gamma} \]  \hspace{1cm} (5)

where

\[ 0 < \frac{1}{\varepsilon} < \gamma = \frac{1}{1 + \beta(\varepsilon - 1)} < 1 \]  \hspace{1cm} (6)

and

\[ h = (1 - \gamma) \left( \frac{\gamma \varepsilon - 1}{w} \right)^{\gamma - 1} (\gamma \varepsilon)^{-\gamma} \]  \hspace{1cm} (7)

Proof: See the Eq. (3) in Abel and Eberly (1999).

2.2 Adjustment Cost Function

Besides production technology and demand conditions, firm’s investment behavior also depends on capital adjustment costs. This section models three forms of adjustment costs that have been highlighted in the investment literature. Abel and Eberly (1994) provide an extensive discussion about the economic rationale of these adjustment costs and the appropriateness of the specification.

2.2.1 Complete and Partial Irreversibility

The early irreversibility literature completely rules out the regime of negative gross investment, hence investment exhibits irreversibility. More recent research allows a wedge between the purchase price of capital \( p^I \) and the sale price of capital \( p^S \), as a result of capital specificity, or more generally, the adverse selection in the second-hand capital goods market. Normalize the purchase price \( p^I \) to one and denote \( b_i = 1 - p^S \geq 0 \), so that the parameter \( b_i \) can be interpreted as the difference between the purchase price and the sale price expressed as a percentage of the purchase price. For example, \( p^S = 0.8 \) gives \( b_i = 0.2 \), indicating that the sale price is 20% lower than the purchase price.

Assumption 5 Irreversibility:

The functional form of irreversibility is

\[ G(I_t) = -b_i I_t 1_{[I_t < 0]} \]

where \( 1_{[I_t < 0]} \) is an indicator equal to one if investment is strictly negative.

Letting \( p^S = 0 \) or \( b_i = 1 \) ensures the firm will never disinvest. This corresponds to the case of complete irreversibility. In contrast, partial irreversibility refers to the more general case where \( 0 < b_i < 1 \).
2.2.2 Fixed Costs

The fixed costs reflect indivisibilities in capital or increasing returns to scale of investment. They are paid at each point of time if any non-zero investment is undertaken. One way to model these costs is to assume them to be proportional to the operating profit.\(^5\) Under this specification, first, these costs can be rationalized as profit loss due to the interruption in production during periods of large adjustment; second, these costs do not become irrelevant as the firm grows larger.

**Assumption 6** **Fixed Costs:**

*The functional form of fixed costs is*

\[
G(X_t, K_t; I_t) = b_f 1_{[I_t \neq 0]} \pi_t
\]

where \(1_{[I_t \neq 0]}\) is an indicator equal to one if investment is non-zero. \(\pi_t\) is defined in equation (5). The parameter \(b_f\) is interpreted as the fraction of operating profit loss due to any non-zero investment.

2.2.3 Quadratic Adjustment Costs

Quadratic adjustment costs reflect those costs that increase convexly in the level of investment. The specification considered here includes three features. First, the costs are quadratic in investment rate, to reflect increasing marginal adjustment cost. Second, the costs attain their minimum value of zero at zero investment, so that the firm can avoid these costs by setting investment equal to zero. Third, the level of these costs is proportional to capital stock, so that a given investment rate imposes costs that increase with the size of the firm, and do not become irrelevant as the firm grows larger.

**Assumption 7** **Quadratic Adjustment Costs:**

*The functional form of quadratic adjustment costs is*

\[
G(K_t; I_t) = b_q \left( \frac{I_t}{K_t} \right)^2 K_t
\]

where \(b_q\) measures the magnitude of quadratic adjustment costs.

The model allows for these three forms of adjustment costs, specifying the adjustment cost function to be

\[
G(X_t, K_t; I_t) = -b_l I_t 1_{[I_t < 0]} + b_f 1_{[I_t \neq 0]} \pi_t + \frac{b_q}{2} \left( \frac{I_t}{K_t} \right)^2 K_t \tag{8}
\]

\(^5\) This specification follows Caballero and Engel (1999) and Bloom (2009). An alternative is to model these fixed costs proportional to the capital stock, such as Caballero and Leahy (1996) and Abel and Eberly (2001), so that \(G(X_t, K_t; I_t) = b_F 1_{[I_t \neq 0]} K_t\), where \(b_F\) is the fraction of capital stock loss due to any non-zero investment. Cooper, Haltiwanger and Power (1999) and Cooper and Haltiwanger (2006) consider both specifications. The later find a model with \(b_F > 0\) fits the investment data better than \(b_F > 0\). That is why this paper focuses on the specification of \(b_f > 0\). For given model parameters specified in Section 3, if \(b_f = 0.05\), similar results for investment policies, short run effects and long run effects are found at around \(b_F = 0.005\).
2.3 Dynamic Optimization

Denote $\Pi(X_t, K_t; I_t)$ as the net revenue of the firm in each period $t$. That is

$$\Pi(X_t, K_t; I_t) = \pi(X_t, K_t; I_t) - G(X_t, K_t; I_t) - I_t$$  \hspace{1cm} (9)

**Assumption 8** The firm is risk-neutral and discounts future net revenue at a constant rate $r$, where $r > \exp(\mu) - 1$.

As explained in Appendix A, the condition $r > \exp(\mu) - 1$ guarantees a finite firm value hence is one of those necessary conditions for the existence of a solution to firm’s optimization problem.

In each period investment is chosen to maximize the discounted present value of current and expected future net revenues, where expectations are taken over the distribution of future demand conditions.

$$V(X_t, K_t) = \max_{I_t} E_t \left\{ \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} \Pi(X_{t+s}, K_{t+s}; I_{t+s}) \right\}$$

According to the Principle of Optimality (Theorem 9.2, Stokey and Lucas, 1989), this investment decision can be represented as the solution to a dynamic optimization problem defined by the stochastic Bellman equation

$$V(X_t, K_t) = \max_{I_t} \left\{ \Pi(X_t, K_t; I_t) + \frac{1}{1+r} E_t [V(X_{t+1}, K_{t+1})] \right\} \hspace{1cm} (10)$$

together with the law of motion (1) and (4) for $K_t$ and $X_t$. Here $V(X_t, K_t)$ is the value of the firm in period $t$; $E_t [V(X_{t+1}, K_{t+1})]$ is the expected value of the firm in period $t + 1$ conditional on information available in period $t$.

2.4 Investment Policy

In the special case of no capital adjustment costs, there is a closed-form solution that describes the optimal investment policy analytically.

2.4.1 Frictionless Case

If $G(X_t, K_t; I_t) \equiv 0$, the Euler equation for the optimization problem (10) is

$$h \left( \frac{X_t}{K_t} \right)^\gamma = J$$  \hspace{1cm} (11)

where

$$J \equiv \frac{r + \delta}{1+r}$$  \hspace{1cm} (12)

The left hand side of equation (11) is the $\text{MPK}$, while the right hand side is known as the Jorgensonian $\text{ucc}$. Hence despite the uncertainty about future demand, this intertemporal optimality condition is equivalent to the first order condition in a static decision problem of the neoclassical producer theory. This is solely the result of the firm being able to adjust its capital stock instantaneously and costlessly.
Proposition 1 Investment Policy in the Frictionless Case:

The optimal frictionless investment rate is

\[ \left( \frac{I_t}{K_t} \right)^* = H \frac{X_t}{K_t} - 1 \] (13)

The optimal frictionless productive capital stock is

\[ \widehat{K}_t^* = I_t^* + K_t = H X_t \] (14)

where

\[ H = \left( \frac{h}{J} \right)^{\frac{1}{2}} \] (15)

Proof: By investment Euler equation.

Equations (13) and (14) imply that without any friction, the optimal investment rate is a linear function of demand relative to inherited capital stock to meet the imbalance between the optimal productive capital stock and the level of demand in each period, where the slope term \( H \) reflects production technology, demand elasticity, factor price, and the Jorgensonian \( ucc \).

2.4.2 Friction Cases

In the presence of general capital adjustment costs specified in equation (8), there is in general no analytical solution to the dynamic optimization problem (10). Appendix A explains how numerical dynamic programming methods are employed to solve such problem.

The investment model outlined above is fully parametric. Sections 2 and 3 impose common parameter values as those in Fig. 1 of Abel and Eberly (1999). That is, depreciation rate \( \delta = 0 \), discount rate \( r = 0.05 \), capital share \( \beta = 0.33 \), demand elasticity \( \varepsilon = 10 \), and demand growth rate \( \mu = 0.029 \). Given the restriction \( \mu > \frac{1}{2} \sigma^2 \), the highest level of uncertainty that could be considered is \( \sigma \equiv 0.2405 \), which is denoted as the reference level of uncertainty in this model. Figures 1-3 present the investment policies derived from the numerical solutions under different forms of capital adjustment costs and at half of the reference level of uncertainty \( \sigma = 0.5 \sigma \).

By plotting the optimal investment rate \( \frac{I_t}{K_t} \) against the scaled demand \( H \frac{X_t}{K_t} - 1 \), the frictionless investment policy is a 45° line. This line is plotted as a benchmark in each of these figures. Since the scaled demand is a monotonic increasing transformation of the \( MPK \), this 45° line highlights the proposition that in the absence of any adjustment cost, investment rate is a continuous and strictly increasing function of the \( MPK \).

Figure 1a illustrates the investment policy with complete irreversibility only \( (b_i = 1.0, b_q = b_f = 0) \) that has been studied in Abel and Eberly (1999), and Figure 1b with partial irreversibility only \( (b_i = 0.10, b_q = b_f = 0) \). In both these two figures, there is a region of inaction in the investment policy. Positive investment is triggered only when the \( MPK \) reaches a right critical level; and at further higher levels of the \( MPK \) the investment rate continues to be lower than what would be chosen in the frictionless case. Since the investment rate on the 45° line would equalize the Jorgensonian \( ucc \) and the
MPK, this implies the introduction of irreversibility increases the ucc relative to the Jorgensonian ucc, as highlighted in Abel and Eberly (1999). Under partial irreversibility, no disinvestment occurs unless the MPK falls below a left critical level; and for further lower levels of the MPK the disinvestment rate is much smaller than what would be chosen in the frictionless case. Under complete irreversibility, no disinvestment would ever happen, no matter how low the MPK is. To summarize, the optimal investment policy under irreversibility is a ‘barrier control’ policy and a non-decreasing function of the MPK.

Figure 2 illustrates both a region of inaction and discontinuities in the investment policy with fixed adjustment costs only \((b_f = 0.05, b_i = b_q = 0)\). Similar to partial irreversibility, investment and disinvestment occur only when the MPK exceeds the right and left critical levels that determine a region of inaction. Outside this region, the optimal investment decisions are quite different from those under partial irreversibility. Small adjustments to the capital stock do not generate benefits that are sufficiently high to warrant paying a fixed cost to implement them. Therefore capital stock adjusts to new information about demand through infrequent but large adjustments. When the MPK exceeds the critical levels, optimal investment rate jumps discontinuously to an investment policy, in which the absolute magnitude is close to or even larger than that in the frictionless case, as the result of two countervailing effects. On the one hand, similar to irreversibility, the introduction of fixed costs increases the ucc relative to the Jorgensonian ucc. Hence the investment rate that equalizes the ucc and the MPK would be lower than the 45° line. On the other hand, as illustrated in Cooper, Haltiwanger and Power (1999), with deterministic positive demand growth in this model (and/or physical capital depreciation more generally) the fixed costs of adjustment provides an incentive for the firm to ‘overshoot its target’, that is whenever investment is implemented, it is optimal to overinvest to make the MPK lower than the ucc. To summarize, the optimal investment policy under fixed adjustment costs is a ‘jump control’ policy and a non-decreasing function of the MPK.

Figure 3 illustrates the optimal investment policy with quadratic adjustment costs only \((b_q = 0.50, b_i = b_f = 0)\). Similar to the frictionless case, with quadratic adjustment costs, investment or disinvestment takes place at all levels of the MPK. However, different from the frictionless case, the rate of adjustment is much smaller than what would be chosen in the frictionless case. This is because the increasing marginal adjustment costs penalize high rates of investment and disinvestment. Capital stock thus adjusts to new information about demand through a series of continuous but small adjustments. To summarize, the optimal investment policy under quadratic adjustment costs is also a continuous and strictly increasing function of the MPK, but much dampened compared with that in the frictionless case.

3 The Effects of Adjustment Costs and Uncertainty

This section examines the effects of uncertainty on the capital stock adjustment and the expected capital stock level under different forms of adjustment costs. To isolate the
effect of uncertainty, the analyses focus on changes in the distribution of demand shocks that preserve the mean level of demand $E [X_t]$.

Lemma 2 Mean-preserving Spread:

Keeping $\mu$ constant and increasing $\sigma$ is a mean-preserving spread for $X_t$, i.e. conditioning on $x_0 = 0$,

$$E [X_t] = \exp (\mu t)$$
$$\text{Var} [X_t] = [\exp (2\mu t)] [\exp (\sigma^2 t) - 1]$$

Proof: By Assumption 4.

3.1 Investment Policies at Different Level of Uncertainty

To study the effects of uncertainty, it is useful to illustrate how investment policies under different forms of adjustment costs would vary with the level of uncertainty. In addition to the investment policies plotted at $\sigma = 0.5\bar{\sigma}$ as those in Figures 1-3, Figures 4-6 add the investment policies at $\sigma = \bar{\sigma}$ on the same figures, keeping all other parameters constant. The comparison between the dark and light lines in Figures 4-6 therefore show the effects of uncertainty on the investment policy under each form of adjustment costs.

Figure 4a and 4b consider these effects under complete irreversibility and partial irreversibility. In both cases, higher level of uncertainty has two effects: first, to enlarge the region of inaction; and second, to lower the rate of positive investment, if positive investment would take place under both levels of uncertainty. This implies the $\text{ucc}$ in the presence of irreversibility is an increasing function of uncertainty, a proposition demonstrated in Abel and Eberly (1999). Similar effects are found in Figure 5 for fixed adjustment costs as well. However, these effects are different in Figure 6 for quadratic adjustment costs, where the shape of investment policy does not vary with the level of uncertainty, but a lower level of uncertainty implies a higher rate of investment for any given level of $MPK$.

3.2 Short-run Capital Stock Adjustment

Following Bloom (2009), this section illustrates the effects of adjustment costs and uncertainty on short-run investment dynamics by considering the impact effect of demand shocks $\epsilon_t$ on the adjustment of the capital stock in the same period. One measure for how much capital stock is adjusted in period $t$ is the change in the log of capital stock level in this period. Denote this measure as $\iota (t)$. That is

$$\iota (t) \equiv \Delta \ln \bar{K}_t = \ln \bar{K}_t - \ln \bar{K}_{t-1}$$

Together with the capital accumulation formula (1), this quantity is approximately equal to investment rate net of depreciation rate, i.e. $\iota (t) = \ln \left[ \left( 1 + \frac{I_t}{\bar{K}_t} \right) (1 - \delta) \right] \simeq \frac{I_t}{\bar{K}_t} - \delta$. A weaker impact effect indicates a smaller response of capital stock to new information about demand, hence slower investment dynamics.
Lemma 3  **Capital Stock Adjustment in the Frictionless Case:**

If \( G(X_t, K_t; I_t) \equiv 0 \), the capital stock adjusts to demand shocks instantaneously and fully according to a one-to-one linear relationship

\[
i^* (t) = \tilde{\mu} + \epsilon_t
\]

Proof: By Proposition 1, Assumption 4 and equation (16).

Figures 7-9 illustrate how the level of uncertainty affects this impact effect under different forms of adjustment costs, at \( \sigma = \bar{\sigma} \) and \( \sigma = 0.5 \bar{\sigma} \). By plotting the capital stock adjustment \( i(t) \) against the demand shocks \( \epsilon_t \), the relationship in the frictionless case is a 45° line. This line is plotted as a benchmark in each of these figures. Since \( \tilde{\mu} = \mu - \frac{1}{2} \sigma^2 \), keeping \( \mu \) constant and varying \( \sigma \) implies that \( \tilde{\mu} \) would vary with the level of uncertainty, which is reflected in the difference between the dash and solid straight lines. However, uncertainty only makes the difference in the intercept but not in the shape or slope of how capital stock responses to demand shocks. Therefore in the absence of adjustment costs, the impact effect of demand shocks on capital stock adjustment is insensitive to the level of uncertainty.

With adjustment costs, Appendix C explains how other curves in Figures 7-9 are simulated using numerical methods, so that comparison between the circle/asterisk lines and the dash/solid 45° lines illustrates the effect of adjustment costs; and comparison between the circle line and asterisk line illustrates the effect of uncertainty.

Figure 7a and 7b consider these effects under complete irreversibility and partial irreversibility. As expected from the investment policies shown in Figure 1a and 1b, the impact effect of positive demand shocks on capital stock growth is much weaker under irreversibility than in the frictionless case. Whereas a firm adjust instantaneously and fully to new information about demand in the frictionless case, if the demand shock leaves a firm within its region of inaction, capital stock does not adjust at all in the current period under irreversibility. If a firm does some adjustment in the current period, the magnitude of the adjustment is much smaller than that in the frictionless case. Also as expected, the impact effect of negative demand shocks on capital stock adjustment is much weaker under partial irreversibility, reflecting the greater reluctance of the firm to undertake disinvestment. This impact effect is exactly zero under complete irreversibility, reflecting the no disinvestment constraint.

Consistent with the investment policies at different levels of uncertainty illustrated in Figure 4a and 4b, the asterisk lines shown in Figure 7a and 7b illustrate that the impact effect of positive demand shocks on capital stock growth is noticeably stronger when the firm subject to irreversibility operates in a less uncertain environment, although how capital stock responses to negative demand shocks is less distinguishable.

Figure 8 illustrates these effects under fixed adjustment costs. Similar to the effect of irreversibility, if the demand shock leaves a firm within its region of inaction, capital stock does not adjust at all in the current period under fixed adjustment costs. Different from the effect of irreversibility, if a firm does some adjustment in the current period,

---

6This is a natural result of the unit root process defined in equation (4). In order to keep the mean of \( X_t \) equal to \( \mu \), the mean of \( \Delta \ln X_t \) and hence of \( \Delta \ln \hat{K}_t \) varies with \( \sigma \).
the magnitude of the adjustment is close to or even larger than that in the frictionless case. Similar to that under irreversibility, the impact effect of positive demand shocks on capital stock growth is also much stronger when the firm subject to fixed adjustment costs operates in a less uncertain environment.

Figure 9 shows these effects under quadratic adjustment costs. As expected from the investment policy shown in Figure 3, the impact effect of both positive and negative demand shocks on capital stock adjustment is much weaker under quadratic adjustment costs than in the frictionless case. Furthermore, consistent with the investment policies at different level of uncertainty illustrated in Figure 6, the impact effect is insensitive to the level of uncertainty over the whole range of demand shocks. Similar to that in the frictionless case, uncertainty only makes the difference in the intercept but not in the shape or slope of how capital stock responses to demand shocks under quadratic adjustment costs.

Properties illustrated in Figures 7-9 are summarized in Proposition 2.

Proposition 2 The Short-run Effect of Adjustment Costs:
If \( b_i > 0 \) or \( b_q > 0 \), \( \partial \ell (t) / \partial e_t < \partial \ell^*(t) / \partial e_t, \forall e_t \);
if \( b_f > 0 \), the effect of \( \partial \ell (t) / \partial e_t \) relative to \( \partial \ell^*(t) / \partial e_t \) is ambiguous.

The Short-run Effect of Uncertainty:
If \( b_i > 0 \) or \( b_f > 0 \), \( \partial^2 \ell (t) / \partial e_t \partial \sigma \leq 0, \forall e_t > 0 \);
if \( b_q > 0 \), \( \partial^2 \ell (t) / \partial e_t \partial \sigma = 0, \forall e_t \).

3.3 Long-run Capital Stock Accumulation

Following Abel and Eberly (1999), this section illustrates the effects of adjustment costs and uncertainty on capital stock accumulation by considering the expected capital stock level \( E \left[ \hat{K}_t \right] \) at different levels of uncertainty \( \sigma \).

Lemma 4 Expected Capital Stock Level in the Frictionless Case:
If \( G(X_t, K_t; I_t) \equiv 0 \), the expected capital stock level is given by

\[
E \left[ \hat{K}_t \right] = H \exp(\mu t) \tag{18}
\]

Proof: By Proposition 1 and Lemma 2.

Following the Eq. (14a) in Abel and Eberly (1999), define \( \kappa(t) \) as the ratio of the expected capital stock level at date \( t \) under different forms of adjustment costs to the expected capital stock level at date \( t \) in the frictionless case. That is

\[
\kappa(t) \equiv \frac{E \left[ \hat{K}_t \right]}{E \left[ \hat{K}_t^* \right]} \tag{19}
\]

Lemma 4 implies the denominator in \( \kappa(t) \) is invariant to the level of uncertainty and is a constant for given parameter values and date \( t \). Therefore how \( \kappa(t) \) is different from 1 reflects the effect of adjustment costs and how \( \kappa(t) \) varies with \( \sigma \) reflects the effect of uncertainty.
Figure 10 is an analytical replicate for the Fig. 1. in Abel and Eberly (1999) and is plotted according to their analytical solution derived in the particular case: complete irreversibility only, no depreciation, infinite time horizon and continuous time.

Figures 11-13 illustrate how $\kappa(t)$ varies with $\sigma$ under different forms of adjustment costs, over a range from a low level of uncertainty $\sigma = \sigma = 0.0485$ to approximate complete certainty to the reference level of uncertainty $\sigma = \bar{\sigma}$. Appendix C explains how these figures are simulated using numerical methods.

Figure 11a plots the ratio $\kappa(t)$ against $\sigma$ with complete irreversibility. Therefore it is a numerical replicate for the Fig 1. in Abel and Eberly (1999) or for Figure 10. The dashed line shows the actual estimates of $\kappa(t)$ at different levels of $\sigma$, which fluctuate somewhat as the result of numerical discretization. The solid line fits a simple 3-order polynomial regression through these points to illustrate the general pattern. This reproduces the main features of Figure 10, which confirms the analytical results in Abel and Eberly (1999) and suggests that our numerical results are in the right ballpark.

There are two key features of $\kappa(t)$ in this special case, which highlight the two important findings from Abel and Eberly (1999). First, $\kappa(t)$ may be greater than, less than, or equal to 1. Second, the behavior of $\kappa(t)$ is not monotonic in the level of uncertainty. To be more specific, at very low levels of uncertainty, the presence of complete irreversibility has almost no effect on the expected level of the capital stock. Indeed as $\sigma = \sigma$, complete irreversibility becomes irrelevant for a firm that is experiencing certain, positive growth in demand. As $\sigma$ increases, $E \left[ \hat{K}_t \right]$ initially increases relative to $E \left[ \hat{K}_t^* \right]$. Over this range the ‘hangover’ effect described in Abel and Eberly (1999) dominates the ‘user cost’ effect, so that on average $\kappa(t) > 1$ and $\partial \kappa(t)/\partial \sigma > 0$. This effect peaks at values of $\sigma$ around 0.17, where $E \left[ \hat{K}_t \right]$ is about 1.3% higher than $E \left[ \hat{K}_t^* \right]$. After this peak, $\partial \kappa(t)/\partial \sigma < 0$. For higher values of $\sigma$, the ‘user cost’ effect dominates the ‘hangover’ effect so that $\kappa(t) < 1$. At $\sigma = \bar{\sigma}$, $E \left[ \hat{K}_t \right]$ is about 0.3% lower than $E \left[ \hat{K}_t^* \right]$.

Figure 11b considers partial irreversibility. The relationship between $\kappa(t)$ and $\sigma$ has a similar pattern to that shown under complete irreversibility, but the magnitudes are different. At low levels of uncertainty, $\kappa(t)$ is again increasing in $\sigma$. At the peak $E \left[ \hat{K}_t \right]$ is about 0.5% higher than $E \left[ \hat{K}_t^* \right]$, and this peak occurs at lower values of $\sigma$ around 0.13. At higher levels of uncertainty, $\kappa(t)$ is again decreasing in $\sigma$. At $\sigma = \bar{\sigma}$, the effect of uncertainty is to reduce $E \left[ \hat{K}_t \right]$ by about 7% of $E \left[ \hat{K}_t^* \right]$. The ‘hangover’ effect appears to be less important under partial irreversibility, where the firm can choose to adjust capital stock downwards, which is ruled out under complete irreversibility.

Figure 12 presents the relationship with fixed adjustment costs. First, different from that under irreversibility, $\kappa(t)$ is less than 1 at complete certainty in the presence of fixed adjustment costs. A firm with deterministic positive demand growth in this model (and/or with physical capital depreciation more generally) will want to have growing capital stock, which requires positive investment on average. Under fixed adjustment costs, this adjustment will take the form of infrequent, large investments, implying a $ucc$ associated with fixed adjustment costs that is higher than the Jorgensonian $ucc$. This user cost effect reduces $E \left[ \hat{K}_t \right]$ relative to $E \left[ \hat{K}_t^* \right]$ by 4% at $\sigma = \sigma$.
Second, under fixed adjustment costs, as illustrated in Figure 5, a higher level of uncertainty will first, enlarge the region of investment inaction. This will reduce both investment and disinvestment relative to that under a lower level of uncertainty, hence has an ambiguous effect on the expected capital stock level; Second, outside the region of inaction, a higher level of uncertainty will decrease the magnitude of investment and increase the magnitude of disinvestment relative to that under a lower level of uncertainty. This will unambiguously reduce the expected capital stock level. Finally, as illustrated in Figure 8, a higher level of uncertainty will enlarge the support of demand shocks, so that some larger capital adjustment which would not occur under a lower level of uncertainty will take place under a higher level of uncertainty. However, since this implies larger adjustment both upwards and downwards, it also has an ambiguous effect on the expected capital stock level. Taking into account all these effects, how $\kappa(t)$ varies with $\sigma$ under fixed adjustment costs is ambiguous. Whether $\kappa(t)$ is larger or smaller than 1 when $\sigma > 0$ is therefore also ambiguous. For the case under illustration, there is an inverse U-shape relationship between $\kappa(t)$ and $\sigma$. At $\sigma = \sigma^*$, the effect of uncertainty is to reduce $E \left[ \hat{K}_t \right]$ by about 6% of $E \left[ \hat{K}_t^* \right]$. This is similar to the magnitude found in the specification with partial irreversibility, and considerably larger than the effect under complete irreversibility.

Figure 13 studies a case with quadratic adjustment costs. First, similar to that under fixed adjustment costs, the presence of quadratic adjustment costs makes $\kappa(t) < 1$ even in an environment with complete certainty. For example, at $\sigma = \sigma^*$, $E \left[ \hat{K}_t \right]$ is about 5% lower than $E \left[ \hat{K}_t^* \right]$. This is because with complete certainty a firm with deterministic positive demand growth in this model (and/or with capital depreciation) will require positive investment. With the functional form of the quadratic adjustment costs considered here, positive investment implies that some adjustment costs must be paid, hence a $ucc$ associated with quadratic adjustment costs that is higher than the Jorgensonian $ucc$. This user cost effect unambiguously reduces $E \left[ \hat{K}_t \right]$ relative to $E \left[ \hat{K}_t^* \right]$ at complete certainty.$^7$

Furthermore, different from fixed adjustment costs, under quadratic adjustment costs, $\kappa(t)$ falls monotonically with $\sigma$. The magnitude of this effect is also much greater than what has been found with partial irreversibility or fixed adjustment costs. For $\sigma = 0.15$, $E \left[ \hat{K}_t \right]$ is about 10% lower than $E \left[ \hat{K}_t^* \right]$. At $\sigma = \sigma^*$, the effect of uncertainty is to reduce $E \left[ \hat{K}_t \right]$ by about 35% of $E \left[ \hat{K}_t^* \right]$. The intuition for this negative monotonic effect lies in three facts.$^8$ First, as illustrated in Figure 9, a higher level of uncertainty will enlarge the

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$^7$Formally, if $b_q > 0$, the closed-form Euler equation of investment implies that $\hat{K}_t = \left( \frac{b}{J + C_t} \right)^{\frac{1}{2}} X_t$, where $C_t = b_q \left( \frac{h}{K_t} \right) - b_q \left( \frac{1 - \delta}{1 + \delta} \right) E_t \left[ \frac{h^2}{K_{t+1}} \right] - \frac{b_q}{2} \left( \frac{1 - \delta}{1 + \delta} \right) E_t \left[ \frac{h}{K_{t+1}} \right]^2$. When $\sigma = 0$ and $\mu > 0$ (and/or $\delta > 0$), there must be an optimal deterministic investment rate $0 < i < 1$. This simplifies $C_t = C \simeq J \cdot b_q \cdot i > 0$. Recall $\hat{K}_t^* = \left( \frac{h}{J + C_t} \right)^{\frac{1}{2}} X_t$, therefore $\hat{K}_t < \hat{K}_t^*$.

$^8$Formally, if $b_q > 0$, when $\sigma > 0$ and $\mu > 0$ (and/or $\delta > 0$), the closed-form Euler equation of investment implies that $E \left[ \hat{K}_t \right] = E \left[ \left( \frac{b}{J + C_t} \right)^{\frac{1}{2}} X_t \right]$, where $C_t \simeq J \cdot b_q \cdot \frac{1}{K_t}$. 

15
support of demand shocks, which implies larger upwards and downwards adjustment will take place relative to that under a lower level of uncertainty. Second, different from fixed adjustment costs, under which the adjustment cost incurred is independent of the rate of investment, the cost incurred under quadratic adjustment costs increases monotonically with the rate of investment. Therefore the user cost effect associated with quadratic adjustment costs increases monotonically with the level of uncertainty. Finally, there is decreasing marginal return to capital \((0 < \gamma < 1)\). This leads to the unambiguous negative relationship between \(E\left[\hat{K}_t\right]\) and \(\sigma\), hence between \(\kappa(t)\) and \(\sigma\).

### 3.4 The Cost-to-Profit Ratio

Compared with \(b_i = 0.1\), which can be interpreted as capital being sold at a price 10% lower than the purchase price if disinvestment occurs, it is less clear how costly capital adjustment is due to a fixed adjustment cost at the magnitude of \(b_f = 0.05\) and a quadratic adjustment cost at the magnitude of \(b_q = 0.50\). The actual adjustment costs incurred as a ratio of the operating profit provides an indication of the relative magnitude of these costs.

When \(\sigma\) increases from \(\sigma\) to \(\bar{\sigma}\), at \(b_f = 0.05\), this cost-to-profit ratio increases monotonically from 0.31% to 0.65%, with an average at 0.38%; at \(b_q = 0.50\), the cost-to-profit ratio increases monotonically from 0.36% to 1.56%, with an average at 0.79%. This implies the actual adjustment costs incurred in the presence of fixed and quadratic adjustment costs both increase with the level of uncertainty.

A related question is why the effect of uncertainty on \(\kappa(t)\) appears to be much larger in the presence of quadratic adjustment costs than that of fixed adjustment costs, at least in the cases illustrated in Figure 13 and 12. Is it simply because that \(b_q = 0.50\) implies a higher average cost-to-profit ratio than that implied by \(b_f = 0.05\), or is it because the effects of uncertainty in the presence of these two forms adjustment costs are fundamentally different, even if they would incur the same cost-to-profit on average?

In order to control for the first possibility, exercises are done to gradually increase the value of \(b_f\) and decrease the value of \(b_q\). At \(b_f = 0.10\), the cost-to-profit ratio increases monotonically from 0.48% to 0.90%, with an average at 0.6%; at \(b_q = 0.30\), the cost-to-profit ratio increases monotonically from 0.23% to 1.25%, with an average at 0.6%, too. At the median/mean level of uncertainty \(\sigma = 0.1445\), the cost-to-profit ratio is about 0.55% for both \(b_f = 0.10\) and \(b_q = 0.30\).

Figure 14 plots how \(\kappa(t)\) varies with \(\sigma\) at \(b_f = 0.10\) and \(b_q = 0.30\) on the same scale. The line for \(\kappa(t)\) associated with \(b_f = 0.10\) starts with 0.94 at \(\sigma = \bar{\sigma}\) and decreases to 0.91 at \(\sigma = \bar{\sigma}\); while the line for \(\kappa(t)\) associated with \(b_q = 0.30\) decreases from 0.97 to 0.79. And it is also around the median/mean level of uncertainty that these two lines intersect. Finally, if the lines for \(\kappa(t)\) at \(b_f = 0.05\) and \(b_q = 0.50\) are added on the same figure, the line for \(b_f = 0.10\) is below the one for \(b_f = 0.05\) at any level of uncertainty; and the line for \(b_q = 0.30\) is above the one for \(b_q = 0.50\) at any level of uncertainty.

This exercise implies first: in the presence of both fixed and quadratic adjustment costs, the cost-to-profit ratio is an informative indicator for how much the adjustment costs would reduce \(E\left[\hat{K}_t\right]\) relative to \(E\left[\hat{K}_t\right]\). Second, \(\kappa(t)\) responses to \(\sigma\) in a stronger
pattern in the presence of quadratic adjustment costs than that of fixed adjustment costs, even if the costs incurred are similar on average over the range of uncertainty. Third, $\kappa(t)$ is a decreasing function of $b_f$ and $b_q$. In other words, all else being equal, higher fixed and quadratic adjustment costs imply lower expected capital stock level.

Suppose $\mu > 0$ (and/or $\delta > 0$), properties discussed in this section are summarized in Proposition 3.

**Proposition 3 The Long-run Effect of Adjustment Costs:**

If $\sigma = 0$, $\kappa(t) \begin{cases} = 1 & \text{if } b_i > 0 \\ < 1 & \text{if } b_f > 0 \text{ or } b_q > 0 \end{cases}$.

If $\sigma > 0$, $\kappa(t) \begin{cases} < 1 & \text{if } b_q > 0 \\ \geq 1 & \text{if } b_i > 0 \text{ or } b_f > 0 \end{cases}$.

Furthermore, $\partial \kappa(t)/\partial b_i > 0$, $\partial \kappa(t)/\partial b_f < 0$, $\partial \kappa(t)/\partial b_q < 0$.

**The Long-run Effect of Uncertainty:**

If $b_q > 0$, $\partial \kappa(t)/\partial \sigma < 0$;

if $b_i > 0$ or $b_f > 0$, the sign of $\partial \kappa(t)/\partial \sigma$ is ambiguous.

4 The Effects of Other Model Parameters

These results are obtained by using particular parameter values imposed in Abel and Eberly (1999). This section studies for given level of uncertainty and capital adjustment costs considered in this model, whether and how these findings would vary with the value of other model parameters.

4.1 Firm’s Characteristics and Economic Environment

Following the section 5 of Abel and Eberly (1999), parameters of interest here are demand growth rate $\mu$, the discount rate $r$, the capital share in the production function $\beta$ and the price elasticity of demand $\varepsilon$.

One could study how the investment policy, short-run capital adjustment and long-run capital accumulation vary with each of these parameters, under each form of capital adjustment costs. For most of these parameters, the variation is found to be most informative in the long-run capital accumulation. Therefore Figures 15-17 focus on the long-run effects only. The lines labelled as ‘AE parameters’ in these figures are plotted at those parameter values imposed in Abel and Eberly (1999) and employed in Figures 11-13, namely $\mu = 0.029$, $r = 0.05$, $\beta = 0.33$, and $\varepsilon = 10$. Using these values as benchmark and varying each of them individually in plotting other four lines provides comparative statics in these figures. The alternative values considered are $\mu = 0.04$, $r = 0.10$, $\beta = 0.13$, and $\varepsilon = 20$.

Abel and Eberly (1999) find that in the special case of complete irreversibility, changing in these parameters leads to clear changes in capital accumulation. To be more specific, first, $\partial \kappa(t)/\partial \mu < 0$. This is because although the irreversibility constraint become less important in a higher growth environment, the hangover effect is weakened even more than the user cost effect as $\mu$ increases. Second, $\partial \kappa(t)/\partial r > 0$. While the user cost under both irreversibility and frictionless case rises with $r$, which tends to reduce the capital
stock level in both cases, this effect is weaker under irreversibility than in the frictionless case. Finally, \( \partial \kappa(t)/\partial \beta < 0 \) and \( \partial \kappa(t)/\partial \varepsilon < 0 \). Since \( \gamma = \frac{1}{1+\delta(\varepsilon-1)} \), as derived in equation (6), together, the capital share and the demand elasticity determine the concavity of the profit function, measured by \( \gamma \). As \( \gamma \) rises, the profit function becomes more concave and thus deviations from the optimal frictionless capital stock are more costly to the firm, so that \( \partial \kappa(t)/\partial \gamma > 0 \), or equivalently \( \partial \kappa(t)/\partial \beta < 0 \) and \( \partial \kappa(t)/\partial \varepsilon < 0 \). Figure 15a presents how \( \kappa(t) \) varies with these four parameters in the presence of complete irreversibility, which confirms above predictions.

Different from complete irreversibility, in the presence of other forms of adjustment costs, since there are no analytical results to draw on, the results illustrated in Figures 15b-17 are based on simulation.

### 4.2 Depreciation Rate

Concerning the effect of depreciation rate, lines for \( \delta = 0.05 \) are added in Figures 15-17 to compare with the benchmark case \( \delta = 0 \). An increase in \( \delta \) makes the investment policy under partial irreversibility more similar to that under complete irreversibility. For example, keeping all other parameters constant, with 5% depreciation rate, even the partial irreversibility is at the magnitude of \( b_i = 0.1 \), no disinvestment would ever happen as if \( b_i = 1 \). This is because with a higher depreciation rate, it is optimal for the firm to sell capital much less often. This ‘less necessary to disinvest’ enhances the ‘hangover’ effect under partial irreversibility, but does not affect the ‘hangover’ effect under completely irreversibility. Meanwhile, a higher \( \delta \) unambiguously increases the Jorgensonian \( \text{ucc} \) in both cases. Therefore \( \partial \kappa(t)/\partial \delta < 0 \) if \( b_i = 1 \) and \( \partial \kappa(t)/\partial \delta > 0 \) if \( b_i = 0.1 \), as illustrated in Figure 15a and 15b.

In the presence of fixed adjustment costs, with a higher \( \delta \), it is optimal for the firm to adjust capital stock more often than otherwise, therefore paying more adjustment costs on average. This user cost effect implies \( \partial \kappa(t)/\partial \delta < 0 \) if \( b_f > 0 \).

In the presence of quadratic adjustment costs, instead of simply affecting the level of \( \kappa(t) \), a higher \( \delta \) will also dampen the effect of uncertainty on \( \kappa(t) \). This is because the user cost associated with quadratic adjustment increases with the rate of investment, while there are two determinants for the optimal investment rate in the presence of quadratic adjustment costs: one is the non-stochastic target level, which is an increasing function of \( \delta \) as demonstrated in Proposition 1; another is the stochastic optimal response to the actual realization of demand shocks. An increase in \( \delta \) increases the user cost by increasing the target investment rate. Meanwhile a higher \( \delta \) also reduces the relative weight of the stochastic part in determining the optimal investment rate and thus the user cost, therefore reduce the sensitivity of \( \kappa(t) \) to the level of uncertainty.

### 4.3 Trend-Stationary Stochastic Process

Finally, in order to allow for the demand shocks to have a persistent but not permanent effect on investment behavior, this section considers an alternative specification for the stochastic process.
**Assumption 9 Alternative Demand Stochastic:**

The law of motion for $X_t$ is

$$
\begin{align*}
    x_t &= \log X_t \\
    x_t &= c + \mu t + \zeta_t \\
    \zeta_t &= \rho \zeta_{t-1} + e_t = \rho^t \zeta_0 + \sum_{s=0}^{t-1} \rho^s e_{t-s}
\end{align*}
$$

(20)

where $0 < \rho < 1$, $e_t = \sigma \epsilon_t$, $\epsilon_t \overset{i.i.d.}{\sim} N(0, 1)$ and $\zeta_0 = 0$.

Under this trend-stationary specification, by adjusting the constant term $c$ properly, keeping $\mu$ constant and increasing $\sigma$ will be a mean-preserving spread for $X_t$ as well.

**Lemma 5 Alternative Mean-preserving Spread:**

For any $0 < \rho < 1$, and $t \to \infty$, by choosing $c = -0.5\sigma^2 / (1 - \rho^2)$, keeping $\mu$ constant and increasing $\sigma$ is a mean-preserving spread for $X_t$, i.e. conditioning on $\zeta_0 = 0$,

$$
\begin{align*}
    E \left[ X_t \right] &= \exp (\mu t) \\
    \text{Var} \left[ X_t \right] &= \left[ \exp (2\mu t) \right] \left[ \exp \left( \frac{\sigma^2}{1 - \rho^2} \right) \right] - 1
\end{align*}
$$

Proof: By Assumption 9.

By construction, the stationary process (20) therefore implies the same expected value for $X_t$ as predicted in Lemma 2 for the non-stationary process (4). Appendix B explains how the dynamic optimization problem (10) can be solved numerically under this alternative specification. When $0 < \rho < 1$, similar patterns for the investment policy, short-run capital adjustment and long-run capital accumulation are found as those illustrated in previous sections. Finally, the relationships between $\kappa(t)$ and $\sigma$ derived at $\rho = 0.9$ are added in Figures 15-17 to illustrate the effects of adjustment costs and uncertainty on capital accumulation under a trend-stationary stochastic process.

### 5 Conclusions

Investment is one of the most important and volatile components of macroeconomic activity. In the short run, the relationship between uncertainty and investment is central to understanding the business cycle. In the long run, the effect of uncertainty on capital accumulation has significant implications for economic growth and development. This paper offers a unified framework to study how capital adjustment costs and uncertainty affect investment dynamics and capital accumulation.

From complete and partial irreversibility, to fixed and quadratic adjustment costs, a simple investment model could generate very rich implications for different investment behavior according to different forms of adjustment costs. The impact effects of demand shocks on capital adjustment also dramatically differ depending on the form of adjustment costs.

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9 Although (4) can be regarded as the limit case of (20) when $\rho \to 1$, the effects of uncertainty generated from (4) and (20) are not completely comparable in general. This is because for a given $E \left[ X_t \right]$, $\text{Var} \left[ X_t \right]$ is time-invariant under Assumption 9, but will increase with $t$ under Assumption 4.
costs; however, these effects may be amplified, dampened or even reversed when translated into the expected long-run capital stock level. Other parameters in the model would also affect the quantities of interest in a substantial fashion.

These findings therefore highlight the importance of empirical work. To obtain the right sign and quantify the magnitude for the effects of uncertainty, it is therefore important to study which form of capital adjustment costs could explain the investment behavior best, conditioning on a good control for the firm’s characteristics and economic environment. This is consistent with the direction in recent empirical work relying on structural estimation and micro-level data, such as Cooper and Haltiwanger (2006), Eberly, Rebelo and Vincent (2008), Bond, Söderbom and Wu (2008) and Bloom (2009).