Some Stories in Number Theory

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1. Some Introduction

2. The Quest for Fermat’s Last Theorem; 1637 – 1995

3. Of Beauty and Utility in Mathematics
Why doing mathematics?

Some common reasons:

- It’s useful (For what, exactly? I don’t see many applications.)
- It’s good to help you think logically (Really, if that’s so, why not just study logic?)
- It helps you in your future career choices, especially if you want to become scientist and or engineer. (OK, now I want to be an artist. So, can I opt out of math classes now, please? Thanks.)
- (Supply other reasons you have heard here.)
- (And here ...)
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Reasons against doing mathematics?

There are many reasonable positions arguing against the need to do mathematics:

- It’s hardly useful. How many percent of the math that I learn in school can be used in my professional and personal life? Close to zero.
- It’s too difficult for most of us anyway. Let’s just leave it to those crazy enough to torture themselves studying it in depth.
- For many students, math has humiliating and traumatizing effects, lowering their self-esteem and making them scared.
- (Add your arguments here.)
Why do mathematicians do mathematics?

I must tell you now that most of the above reasons are, well, given by those who don’t do math seriously enough :) So, why do mathematicians do math?

- It is **BEAUTIFUL**.
- It is **BEAUTIFUL**.
- It is **beautiful**.
- In some cases and sometimes, mathematics could be useful.
- And when it’s useful, it is almost always **powerfully useful**.
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It’s a matter of Aesthetics

Paul Erdős: If numbers are not beautiful, I don’t know what is.

G. H. Hardy: The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

So, let me tell you an amazing story from Number Theory, one of my favourite areas in mathematics, to show you what doing mathematics means.
The Pythagorean Triple

Definition

A Pythagorean triple \((a, b, c) \in \mathbb{N}\) satisfies \(a^2 + b^2 = c^2\). If \((a, b, c)\) is a Pythagorean triple, then so is \((ka, kb, kc)\) for any \(k \in \mathbb{N}\). A primitive Pythagorean triple is one in which \(a, b\) and \(c\) are coprime.

Theorem

(Euler’s Formula) The following will generate all Pythagorean triples uniquely:

\[
a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad \text{and} \quad c = k \cdot (m^2 + n^2)
\]

where \(m, n, k \in \mathbb{N}\) with \(m > n\), \(m - n\) odd, and \(m, n\) coprime.
The Problem and Fermat’s Notebook
Fermat’s Conjecture (1637):

There are no three positive integers $a$, $b$, and $c$ satisfying

$$a^n + b^n = c^n \text{ for } 2 < n \in \mathbb{N}.$$ 

His original write up says:

*It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.*
Fermat supplied the proof for $n = 4$ by a method called *infinite descent*. After this case is settled, it remains to prove for $n$ odd primes (can you see why?).

Between 1637 and 1839, the conjecture was proven for three odd primes $n = 3$ (first by Euler in 1770), $n = 5$ (independently by Legendre and Dirichlet around 1825), and $n = 7$ (first by Lamé in 1839).

Other cases for small primes become more and more difficult to generalize. New methods need to be developed further.
Sophie Germain (see here, ladies can become great mathematicians too!), early 19th century, developed several novel approaches to prove the conjecture for all exponents. She failed to achieve this but managed to prove the conjecture for a family containing infinitely many primes. Now, these primes are called Sophie Germain Primes.

Ernst Kummer (around 1850) developed the concept of ideal numbers and used it to prove the conjecture for all regular prime numbers but failed to settle the case for the irregular primes, which occurs approximately 39% of the time. To see how close he was to a full solution, the only irregular primes below 100 are 37, 59 and 67.

Using computational methods, by 1993, Fermat’s conjecture had been proven for all primes less than four million.
Around 1980s, several leading mathematicians such as Gerhard Frey, Jean-Pierre Serre and Ken Ribet linked Fermat’s Conjecture to a very difficult problem in algebraic geometry called the Taniyama–Shimura-Weil conjecture, posed around 1955. This laid the foundation to the work leading to the proof of Fermat’s problem.

The first successful proof was released in 1994 by Andrew Wiles, after working on it for about 6 years and formally published in 1995. His original proof (1994) actually contained a hole which was subsequently patched by him and a former student of his named Richard Taylor.
Fermat’s Conjecture is now known as Fermat-Wiles Theorem.

It takes 358 years of combined effort by many of the best mathematicians of their respective eras to solve.

And it is completely USELESS. But,

The unsolved problem stimulated the development of Algebraic Number Theory based mostly on the work of Ernst Kummer in the 19th century, and

It led to the proof of the Modularity Theorem in the 20th century.
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A Little Summary

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And what do we have now?

- Algebraic Number Theory makes high-speed and high-fidelity Wireless Communication possible.
- Thanks us mathematicians whenever you use your smartphone, watch movies on your mobile devices, and skype with your loved ones.
- You don’t believe me? Here is an abstract from a recent workshop: Special Semester on Applications of Algebra and Number Theory, Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, October 14 – December 13, 2013.
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Overview

Algebra and number theory have always been counted among the most beautiful mathematical areas with deep proofs and elegant results. However, for a long time they were not considered that important in view of the lack of real-life applications. This has dramatically changed with the appearance of new topics such as modern cryptography, coding theory, and wireless communication. Computational and applied mathematics depending on deep algebraic and number theoretic methods have become crucial for these and other topics of extremely high current interest.

Nowadays we find applications of algebra and number theory frequently in our daily life. We mention security and error detection for internet banking, check digit systems as the bar code on products in a supermarket, GPS and radar, pricing options at a stock market, and eliminating noise for mobile phones as examples.
To most mathematicians it means:

1. Tackling unsolved problems, seeking, first, correct solutions, then make them beautiful.
2. Once a problem is solved, try to generalize it. Removing constraints, making the set up bigger, moving to higher dimensions, etc.
3. Sharing problems, progress (or lack of it!), collaborating, peer-reviewing each other’s work.
4. Teaching the next generations.
What about Applications?

1. Many working mathematicians do not actively seek for possible applications of their work.
2. The stories, while not always as dramatic and long as that of Fermat’s Conjecture, usually work out similarly.
3. Somehow sometimes in the future someone clever would figure out how to use some of the mathematics we have come to know and apply it for something.
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To sum it up, let me end with this profound quote from Jacques Hadamard, a prominent French mathematician.

Practical application is found by not looking for it, and one can say that the whole progress of civilization rests on that principle.