SKewed CYCLIC CODES OVER $F_4R$

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Abstract. This paper considers a new alphabet set, which is a ring that we call $F_4R$, to construct linear error-control codes. Skew cyclic codes over the ring are then investigated in details. We define a nondegenerate inner product and provide a criteria to test for self-orthogonality. Results on the algebraic structures lead us to characterize $F_4R$-skew cyclic codes. Interesting connections between the image of such codes under the Gray map to linear cyclic and skew-cyclic codes over $F_4$ are shown. These allow us to learn about the relative dimension and distance profile of the resulting codes. Our setup provides a natural connection to DNA codes where additional biomolecular constraints must be incorporated into the design. We present a characterization of $R$-skew cyclic codes which are reversible complement.

1. Introduction. The use of noncommutative rings to construct error control codes has recently been an active research area, initiated by the seminal works of Boucher et al. in [8] and [9] as well as that of Abualrub et al. in [1]. In the first two, Boucher and collaborators generalized the notion of cyclic codes by using generator polynomials in a noncommutative polynomial ring called the skew polynomial ring. They supplied examples of skew cyclic codes with Hamming distances larger than previously best-known linear codes of the same length and dimension. In [1], Abualrub et al. generalized the concept of skew cyclic codes to skew quasi-cyclic codes. They then constructed several new codes with Hamming distances exceeding the Hamming distances of the previously best-known linear codes with comparable parameters.

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Another emerging topic in the studies of error correcting codes is additive codes over mixed alphabets. Borges et al. introduced the class of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in [6]. This class generalizes binary and quaternary linear codes. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code is defined to be a subgroup of $\mathbb{F}_2$. In particular, we study the structure of linear skew cyclic codes over $\mathbb{F}_2$. In this paper, we merge the topic of skew cyclic codes with that of codes over mixed alphabets. In particular, we study the structure of linear skew cyclic codes over the ring $\mathbb{F}_4$ where $\mathbb{F}_4$ is the finite field of four elements and $R = \{a + vb \mid a, b \in \mathbb{F}_4\}$ is the commutative ring with 16 elements with $v^2 = v$. Any codeword $c$ in a skew cyclic code $C$ over $\mathbb{F}_4 R$ has the form $c = (a_0, a_1, \ldots, a_{\beta - 1}, b_0, b_1, \ldots, b_{\beta - 1}) \in \mathbb{F}_4 R^{2 \beta}$.

The followings are our contributions.

1. We show that the dual of a skew cyclic code over $\mathbb{F}_4 R$ is also a skew cyclic code. In fact, skew cyclic codes over $\mathbb{F}_4 R$ are left $R[X, \theta]$-submodules of $R_{\alpha, \beta} = \mathbb{F}_2[X]/(X^n - 1) \times R[X, \theta]/(X^\beta - 1)$.
2. We determine their generator polynomials and establish interesting results that relate these codes to cyclic and quasi-cyclic codes over $\mathbb{F}_4 R$. First, we show that a skew cyclic code over $\mathbb{F}_4 R$ is equivalent to an $\mathbb{F}_4 R$-cyclic code if $\alpha$ and $\beta$ are both odd integers. Second, we establish that if $\alpha$ and $\beta$ are both even integers, then an $\mathbb{F}_4 R$-skew cyclic code $C$ is equivalent to an $\mathbb{F}_4 R$ quasi-cyclic code of index 2.
3. Conditions for skew cyclic codes over $\mathbb{F}_4 R$ to be self-orthogonal are studied.
4. We use the Gray mapping to associate these codes to codes over $\mathbb{F}_4$ of length $\alpha + 2 \beta$ and exhibit a nice relationship between these codes and their images over $\mathbb{F}_4$. The Gray image of any skew cyclic code over $\mathbb{F}_4 R$ is the product of a cyclic code over $\mathbb{F}_4$ of length $\alpha$ and two skew cyclic codes, each of length $\beta$. We supply examples of skew cyclic codes over $\mathbb{F}_4 R$ and their respective Gray images for different lengths.
5. Applications of these codes to DNA computing are included in our treatment.

2. Preliminaries. Let $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$ and $R = \{a + vb \mid a, b \in \mathbb{F}_4\}$ be, respectively, the finite field with four elements and the commutative ring with 16 elements where $v^2 = v$. It is well-known that $R$ is a finite non-chain ring with two maximal ideals $\langle v \rangle$ and $\langle v + 1 \rangle$, making $R/\langle v \rangle$ and $R/\langle v + 1 \rangle$ isomorphic to $\mathbb{F}_4$. The Chinese Remainder Theorem then implies that $R = \langle v \rangle \times \langle v + 1 \rangle$. As was shown in [4], $R$ can be uniquely expressed as $\{a + vb = (b + a)v + a(v + 1) \mid a, b \in \mathbb{F}_4\}$.

Let $A \oplus B = \{(a + b) \mid a \in A, b \in B\}$ and $A \otimes B = \{(a, b) \mid a \in A, b \in B\}$ as defined in [14]. An $\mathbb{F}_4$-linear code of length $n$ is a subspace of $\mathbb{F}_4^n$. A subset $C$ of $\mathbb{F}_4^n$ is a linear code over $R$ if $C$ is an $R$-submodule. Given a linear code $C$ over $R$, let

$C_1 \triangleq \{x + y \in \mathbb{F}_4^n \mid (x + y)v + x(v + 1) \in C \text{ for some } x, y \in \mathbb{F}_4^n\}$

and

$C_2 \triangleq \{x \in \mathbb{F}_4^n \mid (x + y)v + x(v + 1) \in C \text{ for some } y \in \mathbb{F}_4^n\}$.

One can quickly verify that $C_1$ and $C_2$ are linear codes over $\mathbb{F}_4$. In fact, any linear code $C$ over $R$ can be expressed as $C = vC_1 \oplus (v + 1)C_2$. Let $r = a + vb \in R$ and $c = (c_1, c_2, \ldots, c_n) \in C$, i.e., $c_j = a_j + vb_j$ with $a_j, b_j \in \mathbb{F}_4$ for $1 \leq j \leq n$. The $j$-th
Definition 2.4. A nonempty subset $\mathcal{F}$ of $R$ is said to be an $\mathcal{F}$-module under the multiplication $x \cdot y = (x + y)v + x(v + 1)$. Hence, $\mathcal{F}$ can be written in terms of $C_1$ and $C_2$ with
\[ x = a(a_1, a_2, \ldots, a_n) \] and
\[ y = (a + b)(b_1, b_2, \ldots, b_n) + b(a_1, a_2, \ldots, a_n). \]

Definition 2.1. Let an automorphism $\theta$ over $R$ be defined by
\[ \theta : R \mapsto R \text{ sending } a + vb \mapsto a^2 + (v + 1)b^2. \] Restricted to $\mathbb{F}_4$, it interchanges $w$ and $w^2$ while keeping $\{0,1\}$ fixed. Note that our $\theta$ here is equal to the composition of automorphisms $\varphi \circ \theta_1$ in [11]. A subset $C$ of $R^n$ is said to be an $R$-skew cyclic code of length $n$ if two conditions are satisfied.

1. $C$ is an $R$-submodule of $R^n$.
2. If $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ then the skew cyclic shift of $c$ over $R$, denoted by $T_b(c) \triangleq (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2}))$, must also be in $C$.

It is often convenient to associate a vector $a = (a_0, a_1, \ldots, a_{n-1})$ with a polynomial $a(X) := a_0 + a_1X + \ldots + a_{n-1}X^{n-1}$ in indeterminate $X$. This allows for constructions of codes using results from the algebra of polynomial rings.

The next two theorems can be inferred by a slight modification on the corresponding theorems in [11] with $q$ restricted to 4. The respective proof is therefore omitted for brevity.

Theorem 2.2. (From [11, Theorem 3]) Let $C = vC_1 \oplus (v + 1)C_2$ be a linear code over $R$. Then $C$ is an $R$-skew cyclic code if and only if $C_1$ and $C_2$ are skew cyclic codes over $\mathbb{F}_4$.

Theorem 2.3. (From [11, Theorem 5]) Let $C = vC_1 \oplus (v+1)C_2$ be a skew cyclic code of length $n$ over $R$. Let $g_1(X)$ and $g_2(X)$ be the respective generator polynomials of $C_1$ and $C_2$ as $\mathbb{F}_4$-skew cyclic codes. Then $C = (vg_1(X) + (v + 1)g_2(X))$. For any element in $R$, we introduce a new ring homomorphism
\[ \eta : R \mapsto \mathbb{F}_4 \text{ sending } a + vb \mapsto a. \] Let $\mathbb{F}_4R := \{(a, b) : a \in \mathbb{F}_4 \text{ and } b \in R\}$. It is straightforward to verify that $\mathbb{F}_4R$ is an $R$-module under the multiplication
\[ d \ast (a, b) = (\eta(d)a, db) \text{ with } d \in R \text{ and } (a, b) \in \mathbb{F}_4R. \] This extends naturally to $\mathbb{F}_4^nR^\beta$. Let $\boldsymbol{x} = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{\beta-1}) \in \mathbb{F}_4^nR^\beta$, for $\alpha$ and $\beta \in \mathbb{N}$, and $d \in R$. Then
\[ d \ast \boldsymbol{x} = (\eta(d)a_0, \eta(d)a_1, \ldots, \eta(d)a_{n-1}, db_0, db_1, \ldots, db_{\beta-1}). \]

Definition 2.4. A nonempty subset $C$ of $\mathbb{F}_4^nR^\beta$ is called an $\mathbb{F}_4R$-linear code if it is an $R$-submodule of $\mathbb{F}_4^nR^\beta$ with respect to the scalar multiplication in Equation (4).

A nondegenerate inner product between $\boldsymbol{x} = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{\beta-1})$ and $\boldsymbol{y} = (d_0, d_1, \ldots, d_{\alpha-1}, e_0, e_1, \ldots, e_{\beta-1})$ is given by
\[ \langle \boldsymbol{x}, \boldsymbol{y} \rangle = v \sum_{i=0}^{\alpha-1} a_id_i + \sum_{j=0}^{\beta-1} b_je_j \in R. \]
The dual code of an $\mathbb{F}_4 R$-linear code $C$, denoted by $C^\perp$, is also $\mathbb{F}_4 R$-linear and is defined in the usual way as

$$C^\perp := \{ y \in \mathbb{F}_4^2 R^\beta \mid \langle x, y \rangle = 0 \text{ for all } x \in C \}.$$  

Let

$$a(X) = a_0 + a_1 X + \ldots + a_{\alpha-1} X^{\alpha-1} \in \mathbb{F}_4[X]/\langle X^\alpha - 1 \rangle$$

and

$$b(X) = b_0 + b_1 X + \ldots + b_{\beta-1} X^{\beta-1} \in R[X, \theta]/\langle X^\beta - 1 \rangle.$$  

Then any codeword $c = (a_0, a_1, \ldots, a_{\alpha-1}, b_0, b_1, \ldots, b_{\beta-1}) \in \mathbb{F}_4^2 R^\beta$ can be identified with a module element consisting of two polynomials such that

$$c(X) = (a(X), b(X)). \quad (6)$$

This identification gives a one-to-one correspondence between $\mathbb{F}_4^2 R^\beta$ and

$$R_{\alpha, \beta} \triangleq \mathbb{F}_4[X]/\langle X^\alpha - 1 \rangle \times R[X, \theta]/\langle X^\beta - 1 \rangle. \quad (7)$$

Let $r(X) = r_0 + r_1 X + \ldots + r_t X^t \in R[X, \theta]$ and $(a(X), b(X)) \in R_{\alpha, \beta}$. Their product is

$$r(X) \ast (a(X), b(X)) = (\eta(r(X))a(X), r(X)b(X)) \quad (8)$$

where $\eta(r(X)) = \eta(r_0) + \eta(r_1) X + \ldots + \eta(r_t) X^t \in \mathbb{F}_4[X]$. Here, $\eta(r(X))a(X)$ is the usual polynomial multiplication in $\mathbb{F}_4[X]/\langle X^\alpha - 1 \rangle$ while $r(X)b(X)$ is the polynomial multiplication in $R[X, \theta]/\langle X^\beta - 1 \rangle$ where $X(a + vb) = (a^2 + (v+1)b^2)X$.

**Theorem 2.5.** $R_{\alpha, \beta}$ is a left $R[X, \theta]$-module with respect to $\ast$ in Equation (8).

**Proof.** Verifying that the required properties are satisfied over $\mathbb{F}_4[X]/\langle X^\alpha - 1 \rangle$ is easy since we do not have to deal with skewness. Verifying over $R[X, \theta]/\langle X^\beta - 1 \rangle$ is routine, albeit tedious. It suffices to use the facts that $\theta$ is a homomorphism with $\theta^{-1} = \theta$. \hfill $\square$

3. **Generator Polynomials of $\mathbb{F}_4 R$-Skew Cyclic Codes.** This section begins with a formal definition of an $\mathbb{F}_4 R$-skew cyclic code and proposes a method to determine the generator polynomial of any $\mathbb{F}_4 R$-skew cyclic code $C$ in $R_{\alpha, \beta}$. We say that two codes are *equivalent* if one can be obtained from the other by some composition of a permutation of the first $\alpha$ positions, a permutation of the last $\beta$ positions, and multiplication of the symbols appearing in a chosen position by a nonzero scalar.

**Definition 3.1.** An $\mathbb{F}_4 R$-linear code $C$ of length $n = \alpha + \beta$ is said to be $\mathbb{F}_4 R$-skew cyclic if, for any codeword $c = (a_0, a_1, \ldots, a_{\alpha-1}, b_0, b_1, \ldots, b_{\beta-1}) \in C$, its skew cyclic shift $T_\theta(c) \triangleq (a_{\alpha-1}, a_0, \ldots, a_{\alpha-2}, \theta(b_{\beta-1}), \theta(b_0), \ldots, \theta(b_{\beta-2}))$ is also in $C$.

**Theorem 3.2.** Let $C$ be an $\mathbb{F}_4 R$-skew cyclic code of length $n = \alpha + \beta$ such that $\beta$ is an even integer. Then $C^\perp$ is also an $\mathbb{F}_4 R$-skew cyclic code of the same length.

**Proof.** It suffices to show that, for any $x = (a_0, a_1, \ldots, a_{\alpha-1}, b_0, b_1, \ldots, b_{\beta-1}) \in C^\perp$, we have $T_\theta(x) \in C^\perp$. Let $y = (d_0, d_1, \ldots, d_{\alpha-1}, e_0, e_1, \ldots, e_{\beta-1})$ be any codeword in $C$. Then

$$\langle T_\theta(x), y \rangle =$$

$$\langle (a_{\alpha-1}, a_0, \ldots, a_{\alpha-2}, \theta(b_{\beta-1}), \theta(b_0), \ldots, \theta(b_{\beta-2})), (d_0, d_1, \ldots, d_{\alpha-1}, e_0, e_1, \ldots, e_{\beta-1}) \rangle =$$

$$v(a_{\alpha-1}d_0 + a_0d_1 + \ldots + a_{\alpha-2}d_{\alpha-1}) + \theta(b_{\beta-1})e_0 + \theta(b_0)e_1 + \ldots + \theta(b_{\beta-2})e_{\beta-1}.$$
Hence, one only needs to show that
\[ 0 = a_{\alpha-1}d_0 + a_0d_1 + \ldots + a_{\alpha-2}d_{\alpha-1} \] and 
\[ 0 = \theta(b_{\beta-1})e_0 + \theta(b_0)e_1 + \ldots + \theta(b_{\beta-2})e_{\beta-1}. \]

Now, let \( \gamma := \text{lcm}(\alpha, \beta) \). Then \( \gamma \) is an even integer since \( \beta \) is an even integer. Since \( C \) is \( \mathbb{F}_4R \)-skew cyclic, for any \( y \in C \) we have \( T_\gamma(y) = y \) and \( T_\gamma^{-1}(y) = C \). Hence, \( \langle \mathbf{x}, T_\gamma^{-1}(y) \rangle = 0 \). Since \( T_\gamma^{-1}(y) = (d_1, \ldots, d_{\alpha-1}, d_0, \theta(e_1), \ldots, \theta(e_{\beta-1}), \theta(e_0)) \), we then obtain
\[
v \sum_{j=0}^{\alpha-1} a_j d_{(j+1) \mod \alpha} + \sum_{j=0}^{\beta-1} b_j \theta(e_{(j+1) \mod \beta}) = 0.
\]

This implies
\[
0 = a_{\alpha-1}d_0 + a_0d_1 + a_1d_2 + \ldots + a_{\alpha-2}d_{\alpha-1} \quad \text{and} \quad \nn0 = b_{\beta-1} \theta(e_0) + b_0 \theta(e_1) + b_1 \theta(e_2) + \ldots + b_{\beta-2} \theta(e_{\beta-1}).
\]

Applying \( \theta \) to both sides of the last equation yields
\[
\theta(0) = \theta(b_{\beta-1})e_0 + \theta(b_0)e_1 + \theta(b_1)e_2 + \ldots + \theta(b_{\beta-2})e_{\beta-1} = 0,
\]
completing the proof.

\[\text{Theorem 3.3.} \quad \text{A code } C \text{ is } \mathbb{F}_4R \text{-skew cyclic if and only if } C \text{ is a left } R[X, \theta] \text{-submodule of } R_{\alpha, \beta} \text{ under the multiplication } \ast.\]

\[\text{Proof.} \quad \text{Let } c(X) = (a(X), b(X)) \text{ be any codeword of an } \mathbb{F}_4R \text{-skew cyclic code } C. \]

Hence, \( (a_0, a_1, \ldots, a_{\alpha-1}, b_0, b_1, \ldots, b_{\beta-1}) \) and all of it’s \( T_\theta \)-skew cyclic shifts are in \( C \). We associate, for each \( j \in \mathbb{N} \), the polynomial
\[
X^j \ast c(X) = (a_{\alpha-j} + a_{\alpha-j+1}X + \ldots + a_{\alpha-j-1}X^{\alpha-1}, \nn\theta^j(b_{\beta-j}) + \theta^j(b_{\beta-j+1})X + \ldots + \theta^j(b_{\beta-j-1})X^{\beta-1})
\]
with the vector
\[
(a_{\alpha-j}, a_{\alpha-j+1}, \ldots, a_{\alpha-j-1}, \theta^j(b_{\beta-j}), \theta^j(b_{\beta-j+1}), \ldots, \theta^j(b_{\beta-j-1})).
\]
The indices of the first block (of length \( \alpha \)) are taken modulo \( \alpha \) and those of the second block (of length \( \beta \)) are taken modulo \( \beta \). By the \( \mathbb{F}_4R \)-linearity of \( C \), we have \( r(X) \ast c(X) \in C \) for any \( r(X) \in R[X, \theta] \). Thus, \( C \) is a left \( R[X, \theta] \)-submodule of \( R_{\alpha, \beta} \).

Conversely, let \( C \) be a left \( R[X, \theta] \)-submodule of the left \( R[X, \theta] \)-module \( R_{\alpha, \beta} \). Then, for any \( c(X) \in C \), we have \( X^j \ast c(X) \in C \) for any \( j \in \mathbb{N} \). Thus, \( C \) is indeed an \( \mathbb{F}_4R \)-skew cyclic code.

Let \( C \) be an \( \mathbb{F}_4R \)-skew cyclic code. Let \( c(X) = (a(X), b(X)) \) be an element in \( C \). Let \( \ell(X) \) be an element in \( \mathbb{F}_4[X] / (X^\alpha - 1) \). We use \( 0 \) to denote either the zero vector \((0, 0, \ldots, 0)\) or the zero polynomial. Let
\[
I \triangleq \{ b(X) \in R[X, \theta] / (X^\beta - 1) \mid (\ell(X), b(X)) \in C \} \text{ and } 
J \triangleq \{ a(X) \in \mathbb{F}_4[X] / (X^\alpha - 1) \mid (a(X), 0) \in C \}.
\]
The next results establish useful properties of the sets \( I \) and \( J \)

\[\text{Lemma 3.4.} \quad J \text{ is an ideal in } \mathbb{F}_4[X] / (X^\alpha - 1) \text{ generated by a left divisor of } X^\alpha - 1.\]
Lemma 3.5. \( I \) is a principally generated left \( R[X, \theta] \)-submodule of \( R[X, \theta]/\langle X^\beta - 1 \rangle \).

Proof. Let \( b_1(X) \) and \( b_2(X) \) be elements in \( I \). Then there exist polynomials \( \ell_1(X) \) and \( \ell_2(X) \) in \( F_4[X]/\langle X^\alpha - 1 \rangle \) such that \((\ell_1(X), b_1(X)), (\ell_2(X), b_2(X)) \in C \). Hence,

\[
(\ell_1(X), b_1(X)) + (\ell_2(X), b_2(X)) = (\ell_1(X) + \ell_2(X), b_1(X) + b_2(X)) \in C,
\]

implying \( b_1(X) + b_2(X) \in I \). Let \( r(X) \in R[X, \theta]/\langle X^\beta - 1 \rangle \) and \((\ell(X), b(X)) \in C \). Since \( C \) is a left \( R[X, \theta] \)-submodule of \( R_{\alpha, \beta} \), we have

\[
r(X) \ast (\ell(X), b(X)) = (\eta(r(X)))\ell(X) \text{ modulo } (X^\alpha - 1), r(X)b(X) \text{ modulo } (X^\beta - 1)\]

in \( C \), making \( r(X)b(X) \text{ modulo } (X^\beta - 1) \in I \). Thus, \( I \) is a left submodule in \( R[X, \theta]/\langle X^\beta - 1 \rangle \) and, by Theorem 2.3, \( I = \langle g(X) \rangle \) where

\[
g(X) \doteq vg_1(X) + (v + 1)g_2(X). \tag{9}
\]

The following result classifies all \( F_4 \)-skew cyclic codes.

Theorem 3.6. Let \( g(X) \) be as defined in Equation (9). Let \( C \) be an \( F_4 \)-skew cyclic code. Then \( C \) is generated as a left submodule of \( R_{\alpha, \beta} \) by \((f(X), 0)\) and \((\ell(X), g(X))\) where \( \ell(X) \) is an element in \( F_4[X]/\langle X^\alpha - 1 \rangle \) and \( f(X) \) is a left divisor of \( X^\alpha - 1 \).

Proof. Let \( c = (c_1, c_2) \in C \) with \( c_1 \in F_4[X] \) and \( c_2 \in R^2 \). Then \( c_2(X) \in I \) and we write \( c_2(X) = q(X)g(X) \) for some \( q(X) \in R[X, \theta]/\langle X^\beta - 1 \rangle \). There exist \( \ell(X) \in F_4[X]/\langle X^\alpha - 1 \rangle \) such that \((\ell(X), g(X)) \in C \) since \( g(X) \in I \). We have

\[
c = (c_1, c_2) = (c_1(X), 0) + (0, q(X)g(X))
= (c_1(X), 0) + (\eta(q(X))\ell(X), q(X)g(X)) + (\eta(q(X)))\ell(X), 0)
= (c_1(X), 0) + q(X) * ((\ell(X), g(X)) + (\ell(X), 0)).
\]

Hence, \( (\eta(q(X)))\ell(X) + c_1(X), 0 \) \in C, making \( \eta(q(X))\ell(X) + c_1(X) \in J \). By Lemma 3.4, there exists \( p(X) \in J \) satisfying \( \eta(q(X))\ell(X) + c_1(X) = p(X)f(X) \). Thus, \( c(X) = q(X) * (\ell(X), g(X)) + (p(X)f(X), 0) \).

Lemma 3.7. Let \( C \) be an \( F_4 \)-skew cyclic code. Then, without loss of generality, we can assume \( \deg(\ell(X)) < \deg(f(X)) \).

Proof. Suppose that \( \deg(\ell(X)) - \deg(f(X)) = k \geq 0 \). Consider the code \( D \) generated by the set

\[
\{(f(X), 0), (\ell(X), g(X)) + sX^k \ast (f(X), 0)\} = \{(f(X), 0), (\ell_1(X), g(X))\}
\]
where \( \ell_1(X) = \ell(X) + sX^k f(X) \) for some \( s \in F_4 \). Hence, \( D \subseteq C \). On the other hand,

\[
(\ell(X), g(X)) = (\ell(X), g(X)) + sX^k \ast (f(X), 0) - sX^k \ast (f(X), 0).
\]

Hence, \( C \subseteq D \), making \( C = D \). Notice here that \( \deg(\ell_1(X)) < \deg(\ell(X)) \). We repeat the same process on \( \ell_1(X) \) until we obtain \( \deg(\ell(X)) < \deg(f(X)) \).
The second to the last equation is due to \(2\gamma\) since 

**Proof.** Let \(C\) be an \(F_4R\)-skew cyclic code and \(\gamma := \text{lcm}(\alpha, \beta)\). Then \(\text{gcd}(\gamma, 2) = 1\) since \(\gamma\) is odd. Then there exist integers \(k\) and \(j\) such that \(\gamma k + 2j = 1\) and, hence, \(2j = 1 - \gamma k = 1 + \gamma t\) for some \(t > 0\) where \(t \equiv -k \pmod{\gamma}\). As in Equation (6), let 

\[ c(X) = (a(X), b(X)) \in C. \]

Then

\[
X^{2j} \ast c(X) = X^{2j} \ast \left( \sum_{i=0}^{\alpha-1} a_i X^i, \sum_{i=0}^{\beta-1} b_i X^i \right) = \left( \sum_{i=0}^{\alpha-1} a_i X^{i+2j}, \sum_{i=0}^{\beta-1} \theta^{2j}(b_i) X^{i+2j} \right)
\]

\[
= \left( \sum_{i=0}^{\alpha-2} a_i X^{i+1+\gamma t}, \sum_{i=0}^{\beta-1} \theta^{2j}(b_i) X^{i+1+\gamma t} \right)
\]

\[
= \left( \sum_{i=0}^{\alpha-2} a_i X^{i+1+\gamma t} + a_{\alpha-1} X^{\alpha+\gamma t}, \sum_{i=0}^{\beta-2} b_i X^{i+1+\gamma t} + a_{\beta-1} X^{\beta+\gamma t} \right)
\]

\[
= \left( \sum_{i=0}^{\alpha-2} a_i X^{i+1} + a_{\alpha-1}, \sum_{i=0}^{\beta-2} b_i X^{i+1} + b_{\beta-1} \right).
\]

The second to the last equation is due to \(\theta^2(r) = r\) for all \(r \in R\) while the last equation follows because \(X^\alpha = X^\beta = X^\gamma = 1\). \(\square\)

**Theorem 3.9.** An \(F_4R\)-skew cyclic code is equivalent to an \(F_4R\)-quasi-cyclic code of index 2 if both \(\alpha\) and \(\beta\) are even integers.

**Proof.** Let \(C\) be an \(F_4R\)-skew cyclic code, \(\alpha = 2N\), and \(\beta = 2M\) for some \(N, M \in \mathbb{N}\). Then \(\gamma = \text{lcm}(\alpha, \beta)\) is an even integer with \(\text{gcd}(\gamma, 2) = 2\). For any 

\[ c = (a_0, a_1, \ldots, a_{N-1}, a_{N-1}, b_0, b_1, \ldots, b_{M-1}, b_{M-1}) \in C \]

there exist integers \(k \geq 0\) and \(j\) such that \(2j = 2 + k\gamma\). Consider 

\[
T_{\theta^{k+\gamma}}(a_0, a_1, \ldots, a_{N-1}, a_{N-1}, b_0, b_1, \ldots, b_{M-1}, b_{M-1}) = T_{\theta^\gamma}(a_{N-1}, a_{N-1}, \ldots, a_{N-2}, a_{N-2}, b_{M-1}, b_{M-1}, \ldots, b_{M-2}, b_{M-2}) = (a_{N-1}, a_{N-1}, \ldots, a_{N-2}, a_{N-2}, b_{M-1}, b_{M-1}, \ldots, b_{M-2}, b_{M-2}) \in C
\]

since \(T_{\theta^\gamma}(c) = c\) for any \(c \in F_2^\alpha R^\beta\). Thus, \(C\) is equivalent to an \(F_4R\)-quasi cyclic code of length \(n = \alpha + \beta\) and index 2. \(\square\)

**4. The Gray Mapping.** The classical Gray mapping \(\phi^*: R \rightarrow F_4^2\) is defined by 

\(\phi^*(a + vb) = (a + b, a)\) for any \(a + vb \in R\). The Lee weight of any element in \(R\) is the Hamming weight of its image under \(\phi^*\). This map extends naturally to vectors in \(R^n\). For any \(x = (x_0, x_1, \ldots, x_{n-1}) \in F_2^n\) and \(y = (y_0, y_1, \ldots, y_{n-1}) \in R^\beta\), the Gray map over \(F_4R\) is defined by 

\[
\phi: F_4^\alpha R^\beta \rightarrow F_4^{\alpha+2\beta}\ 	ext{ sends } (x, y) \to (x, \phi^*(y)).
\]

The map \(\phi\) is an isometry which transforms the Lee distance in \(F_4^\alpha R^\beta\) to the Hamming distance in \(F_4^{\alpha+2\beta}\). For any \(F_4R\)-linear code \(C\), the code \(\phi(C)\) is \(F_4\)-linear. Furthermore, we have 

\[
wt(x, y) = wt_H(x) + wt_L(y)
\]

where \(wt_H(x)\) is the Hamming weight of \(x\) and \(wt_L(y)\) is the Lee weight of \(y\).
Theorem 4.1. Let $C$ be a self-orthogonal $\mathbb{F}_4R$-linear code under the inner product defined in Equation (5). Then $\phi(C)$ is a Euclidean self-orthogonal code over $\mathbb{F}_4$.

Proof. It suffices to show that the Gray images of codewords are Euclidean orthogonal whenever the codewords are orthogonal. Let $C$ be a self-orthogonal $\mathbb{F}_4R$-linear code of length $\alpha + \beta$. Let $v = (a, b + v e), w = (d, u + v s) \in \mathbb{F}_4^2 \times R^3$ be codewords in $C$ with $a, d, u, s \in \mathbb{F}_4^2$ and $b, c, u, s \in R^3$. Then, by Equation (5),

$$\langle v, w \rangle = v(a \cdot d) + b \cdot u + v(b \cdot s + c \cdot u + c \cdot s) = 0 + v0 \in R.$$ Hence, $b \cdot u = 0$ and $a \cdot d + b \cdot s + c \cdot u + c \cdot s = 0$. Since $\phi(v) = (a, b + c, b)$ and $\phi(w) = (d, u + s, u)$, one gets

$$\phi(v) \cdot \phi(w) = a \cdot d + b \cdot u + b \cdot s + c \cdot u + c \cdot s + b \cdot u = 0.$$ Therefore, the code $\phi(C)$ is Euclidean self-orthogonal. \hfill \square

Theorem 4.2. Let $C$ be an $\mathbb{F}_4R$-skew cyclic code of length $n = \alpha + 2\beta$. Then, $\phi(C) = C_0 \otimes C_1 \otimes C_2$, where $C_0$ is a cyclic code of length $\alpha$ in $\mathbb{F}_4[X]/(X^\alpha - 1)$ and both $C_1$ and $C_2$ are skew cyclic codes of length $\beta$ in $R[X]/(X^\beta - 1)$. Moreover,

$$|\phi(C)| = \prod_{i=0}^{2}|C_i|.$$ Proof. From $\{x = (a_0, a_1, \ldots, a_{\alpha-1}, b_0 + vc_0, b_1 + vc_1, \ldots, b_{\beta-1} + vc_{\beta-1}) : x \in C\}$, we construct the codes

$$C_0 := \{(a_0, a_1, \ldots, a_{\alpha-1})\}, C_1 := \{(b_0 + c_0, b_1 + c_1, \ldots, b_{\beta-1} + c_{\beta-1})\},$$

and $C_2 := \{(b_0, b_1, \ldots, b_{\beta-1})\}$. A codeword $u := (a_0, a_1, \ldots, a_{\alpha-1}) \in C_0$ corresponds to a codeword

$$x = (a_0, a_1, \ldots, a_{\alpha-1}, b_0 + vc_0, b_1 + vc_1, \ldots, b_{\beta-1} + vc_{\beta-1}) \in C.$$ Since $C$ is an $\mathbb{F}_4R$-skew cyclic code, we know that $T_b(x)$ is given by

$$T_b(x) = (a_{\alpha-1}, a_0, a_1, \ldots, a_{\alpha-2}, b_{\beta-1} + vc_{\beta-1}, b_{\beta-2} + vc_{\beta-2}, \ldots, b_0 + vc_0, \ldots, b_{\beta-2} + vc_{\beta-2}) \in C.$$ Hence, $(a_{\alpha-1}, a_0, a_1, \ldots, a_{\alpha-2}) \in C_0$. This implies that $C_0$ is a cyclic code of length $\alpha$ in $\mathbb{F}_4[X]/(X^\alpha - 1)$.

The proof that both $C_1$ and $C_2$ are skew cyclic codes of length $\beta$ in $R[X]/(X^\beta - 1)$ follows the same line of argument. Thus, $\phi(C) = C_0 \otimes C_1 \otimes C_2$ and $|\phi(C)| = \prod_{i=0}^{2}|C_i|$. \hfill \square

Lemma 4.3. Let $C = \langle (f(x), 0), (0, g(x)) \rangle$ be an $\mathbb{F}_4R$-skew cyclic code with $\ell(X) := 0$. Then $C = C_1 \otimes C_2$ where $C_1$ is a skew cyclic code over $\mathbb{F}_4$ and $C_2$ is a skew cyclic code over $R$.

Proof. Note that $c = (c_1, c_2) \in C$ if and only if $c_1 = q_1 f(X)$ and $c_2 = q_2 g(X)$ if and only if $c_1 \in C_1 = (f(X))$ and $c_2 \in C_2 = (g(X))$. Thus, $C = C_1 \otimes C_2$ where $C_1 = (f(X))$ and $c_2 \in C_2 = (g(X))$. \hfill \square

Lemma 4.4. Let $C = C_1 \otimes C_2$ where $C_1$ is an $\mathbb{F}_4$-skew cyclic Euclidean self-orthogonal code and $C_2$ is an $R$-skew cyclic self-orthogonal code over $R$. Then $C$ is a self-orthogonal $\mathbb{F}_4R$-skew cyclic code.
Suppose this respect, with linear codes directly constructed algebraically over the corresponding field. In ring under the Gray map is a code over a field. This latter code is usually inferior particularly as revealed in Theorem 5. Skew cyclic codes were subsequently used as profiles. Therefore, \( C \subseteq C^\perp \), implying that the \( \mathbb{F}_4 R \)-skew cyclic code \( C \) is self-orthogonal.

Note that the converse does not hold. In fact, \( C^\perp \neq C^\perp \otimes C^\perp \) in general.

**Table 1.** Examples, in increasing minimum distances, of good choices of \( \mathbb{F}_4 \)-skew cyclic codes \( C_1 \) and \( C_2 \) to use in Theorem 4.2.

<table>
<thead>
<tr>
<th>No.</th>
<th>Parameter</th>
<th>Generator Polynomial(s) for ( C_1 ) and ( C_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[6,3,4]</td>
<td>( w + w^2X + w^2X^2 + X^3 )</td>
</tr>
<tr>
<td>2</td>
<td>[30,19,7]</td>
<td>( w^2X + X^2 + X^4 + w^2X^6 + X^8 + w^2X^{10} + X^{11} ) and ( w + w^2X + X^2 + X^4 + w^2X^6 + X^8 + w^2X^{10} + X^{11} )</td>
</tr>
<tr>
<td>3</td>
<td>[12,3,8]</td>
<td>( w^2 + wX + wX^2 + X^3 + w^2X^6 + wX^7 + wX^8 + X^9 )</td>
</tr>
<tr>
<td>4</td>
<td>[22,11,8]</td>
<td>( 1 + X + wX^2 + wX^3 + X^4 + X^5 + X^6 + X^7 + w^2X^8 + w^2X^9 + X^{10} + X^{11} )</td>
</tr>
<tr>
<td>5</td>
<td>[38,19,12]</td>
<td>( 1 + X + w^2X^2 + w^2X^3 + w^2X^6 + w^2X^7 + w^2X^8 + w^2X^9 + wX^{10} + wX^{11} + wX^{12} + wX^{13} + wX^{16} + wX^{17} + X^{18} + X^{19} ) and ( 1 + X + wX^2 + wX^3 + wX^6 + wX^7 + wX^8 + wX^9 + w^2X^{10} + w^2X^{11} + w^2X^{12} + w^2X^{13} + w^2X^{16} + w^2X^{17} + X^{18} + X^{19} )</td>
</tr>
<tr>
<td>6</td>
<td>[30,6,18]</td>
<td>( w + wX + w^2X^2 + X^3 + X^5 + w^2X^6 + wX^7 + X^8 + wX^9 + X^{10} + wX^{11} + wX^{12} + X^{14} + X^{15} + w^2X^{16} + wX^{17} + X^{18} + w^2X^{19} + w^2X^{20} + X^{22} + X^{23} + X^{24} )</td>
</tr>
</tbody>
</table>

5. \( \mathbb{F}_4 \)-Codes from \( \mathbb{F}_4 R \)-Skew Cyclic Codes. The image of a code over a given ring under the Gray map is a code over a field. This latter code is usually inferior in terms of the usual measure of relative rate and relative distance when compared with linear codes directly constructed algebraically over the corresponding field. In this respect, \( \mathbb{F}_4 R \)-skew cyclic codes are not exempted. Their excellent structures, particularly as revealed in Theorem 4.2, allow us to determine good choices of the ingredient codes \( C_0, C_1, \) and \( C_2 \) that result in best-possible dimension and distance profiles.

Since factorization in the skew polynomial ring \( \mathbb{F}_4[X, \theta] \) requires considerably more care than in \( \mathbb{F}_4[X] \), we started by finding good codes based on the factorization of \( X^3 - 1 \in \mathbb{F}_4[X, \theta] \). A search for good skew cyclic codes was done following the suggestion of Caruso and Le Borgne in [10]. Through personal communication Le Borgne sent us an implementation routine in MAGMA [7]. The identified good skew cyclic codes were subsequently used as \( C_1 \) or \( C_2 \).

Based on the particular structure described in Theorem 4.2, the three ingredient codes \( C_0, C_1, \) and \( C_2 \) are combined by the \texttt{DirectSum} routine in MAGMA to yield the code \( \phi(C) \). To minimize the drop in the relative dimension and relative distance of \( \phi(C) \), we chose the ingredient codes to have equal minimum distances and
relatively small dimensions. Examples of good choices for the codes $C_1$ and $C_2$ are given in Table 1 while Table 2 contains the cyclic codes that we used as $C_0$. The best parameters of the resulting $\phi(C)$ are listed in Table 3.

6. DNA Skew Cyclic Code over $F_4 R$. The encoding and decoding systems to store or transfer information or data by mimicking DNA sequences are known

<table>
<thead>
<tr>
<th>No.</th>
<th>Parameter</th>
<th>Generator Polynomial of $C_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[4, 1, 1]$</td>
<td>$X^3 + X^2 + X + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$[5, 2, 1]$</td>
<td>$X^3 + wX^2 + wX + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$[7, 3, 1]$</td>
<td>$X^4 + X^2 + X + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$[15, 11, 1]$</td>
<td>$X^4 + X^3 + X^2 + w^2X + w$</td>
</tr>
<tr>
<td>5</td>
<td>$[17, 13, 1]$</td>
<td>$X^4 + X^3 + wX^2 + X + 1$</td>
</tr>
<tr>
<td>6</td>
<td>$[35, 30, 1]$</td>
<td>$X^5 + w^2X^4 + wX^2 + wX + 1$</td>
</tr>
<tr>
<td>7</td>
<td>$[15, 7, 1]$</td>
<td>$X^8 + X^6 + wX^5 + wX^4 + X^3 + wX^2 + w^2$</td>
</tr>
<tr>
<td>8</td>
<td>$[17, 9, 1]$</td>
<td>$X^8 + wX^7 + wX^5 + wX^4 + wX^3 + wX + 1$</td>
</tr>
<tr>
<td>9</td>
<td>$[19, 10, 1]$</td>
<td>$X^9 + wX^8 + wX^6 + wX^5 + w^2X^4 + w^2X^3 + w^2X + 1$</td>
</tr>
<tr>
<td>10</td>
<td>$[35, 24, 1]$</td>
<td>$X^{11} + wX^{10} + X^9 + wX^8 + wX^7 + wX^5 + wX^3 + X + 1$</td>
</tr>
<tr>
<td>11</td>
<td>$[41, 30, 1]$</td>
<td>$X^{11} + wX^{10} + wX^9 + w^2X^6 + w^2X^5 + wX^4 + w^2X^3 + wX + 1$</td>
</tr>
<tr>
<td>12</td>
<td>$[15, 6, 1]$</td>
<td>$X^9 + wX^8 + X^7 + X^5 + wX^4 + w^2X^2 + w^3X + 1$</td>
</tr>
<tr>
<td>13</td>
<td>$[17, 8, 1]$</td>
<td>$X^9 + wX^8 + X^7 + wX^6 + wX^5 + wX^3 + wX^2 + w^2X + 1$</td>
</tr>
<tr>
<td>14</td>
<td>$[19, 9, 1]$</td>
<td>$X^{10} + wX^9 + w^2X^8 + w^2X^7 + X^5 + wX^4 + wX^2 + w^2X + 1$</td>
</tr>
<tr>
<td>15</td>
<td>$[21, 10, 1]$</td>
<td>$X^{11} + wX^{10} + X^8 + w^2X^7 + wX^6 + X^5 + wX^4 + X^3 + wX^2 + X + w$</td>
</tr>
<tr>
<td>16</td>
<td>$[35, 23, 1]$</td>
<td>$X^{12} + w^2X^{11} + w^2X^{10} + wX^9 + wX^7 + wX^6 + wX^5 + wX^4 + wX^3 + X^2 + 1$</td>
</tr>
<tr>
<td>17</td>
<td>$[43, 29, 1]$</td>
<td>$X^{14} + w^2X^{13} + w^2X^{12} + X^{11} + w^2X^9 + wX^5 + X^3 + wX^2 + wX + 1$</td>
</tr>
<tr>
<td>18</td>
<td>$[15, 2, 1]$</td>
<td>$X^{13} + wX^{12} + wX^{11} + X^{10} + X^8 + wX^7 + wX^6 + X^5 + X^3 + wX^2 + wX + 1$</td>
</tr>
<tr>
<td>19</td>
<td>$[17, 4, 1]$</td>
<td>$X^{13} + X^{12} + wX^{11} + X^9 + wX^8 + w^2X^7 + w^2X^6 + wX^5 + X^4 + wX^2 + X + 1$</td>
</tr>
<tr>
<td>20</td>
<td>$[21, 6, 1]$</td>
<td>$X^{15} + X^{14} + w^2X^{13} + w^2X^{12} + X^{11} + wX^{10} + X^9 + w^2X^8 + X^7 + X^6 + X^5 + wX^4 + w^2X^3 + X^2 + wX + 1$</td>
</tr>
<tr>
<td>21</td>
<td>$[29, 14, 1]$</td>
<td>$X^{15} + w^2X^{14} + wX^{13} + wX^{12} + X^{11} + wX^{10} + wX^9 + X^8 + X^7 + wX^6 + wX^5 + X^4 + wX^3 + wX^2 + w^2X + 1$</td>
</tr>
<tr>
<td>22</td>
<td>$[37, 18, 1]$</td>
<td>$X^{19} + wX^{18} + wX^{17} + wX^{15} + X^{13} + w^2X^{12} + wX^{11} + wX^{10} + wX^9 + wX^8 + w^2X^7 + X^6 + wX^4 + wX^2 + wX + 1$</td>
</tr>
<tr>
<td>23</td>
<td>$[39, 19, 1]$</td>
<td>$X^{20} + w^2X^{18} + w^2X^{17} + wX^{15} + X^{14} + X^{12} + X^{11} + X^{10} + wX^9 + w^2X^8 + wX^6 + X^5 + X^3 + wX^2 + w$</td>
</tr>
<tr>
<td>24</td>
<td>$[43, 14, 1]$</td>
<td>$X^{29} + w^2X^{28} + X^{27} + wX^{24} + w^2X^{23} + X^{20} + w^2X^{19} + wX^{17} + w^2X^{15} + wX^{14} + w^2X^{12} + wX^{10} + X^9 + wX^6 + w^2X^5 + X^2 + wX + 1$</td>
</tr>
</tbody>
</table>
The process in which a strand and its complement bind to form a double-helix is known as Watson-Crick complementation. The reverse complement of a strand is the strand obtained by replacing each A by T and vice versa, and each G by C and vice versa. One writes \( \overline{A} = T, \overline{T} = A, \overline{C} = G, \text{ and } \overline{G} = C \). Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \) be distinct codewords in a DNA code \( \mathcal{D} \). The reverse of \( \mathbf{x} \) is \( \mathbf{x}_{\text{rev}} = (x_n, x_{n-1}, \ldots, x_1) \). The complement of \( \mathbf{x} \) is \( \mathbf{x}^c = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}) \). Hence, \( \mathbf{x}_{\text{rev}}^c = (\overline{x_n}, \overline{x_{n-1}}, \ldots, \overline{x_1}) \) is the reverse complement of \( \mathbf{x} \).

The Watson-Crick complementation of a strand is the strand obtained by replacing each A by T and vice versa, and each G by C and vice versa. One writes \( \overline{A} = T, \overline{T} = A, \overline{C} = G, \text{ and } \overline{G} = C \). Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \) be distinct codewords in a DNA code \( \mathcal{D} \). The reverse of \( \mathbf{x} \) is \( \mathbf{x}_{\text{rev}} = (x_n, x_{n-1}, \ldots, x_1) \). The complement of \( \mathbf{x} \) is \( \mathbf{x}^c = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}) \). Hence, \( \mathbf{x}_{\text{rev}}^c = (\overline{x_n}, \overline{x_{n-1}}, \ldots, \overline{x_1}) \) is the reverse complement of \( \mathbf{x} \).

The resulting \( F_4 \)-linear code \( \phi(C) \) has length \( n + k \), dimension \( k_0 + k_1 + k_2 \), and minimum distance \( \delta \).

Table 3. Examples of Good \( \phi(C) \) where \( C \) is an \( F_4R \)-Skew Cyclic Codes of Length \( n + \alpha + \beta \). The construction is based on Theorem 4.2. The ingredient codes are chosen with \( \delta := d(C_0) = d(C_1) = d(C_2) \) and \( \alpha \) is the length of \( C_0 \) while \( \beta \) is the length of \( C_1 \) and \( C_2 \). The resulting \( F_4 \)-linear code \( \phi(C) \) has length \( n + \alpha + \beta \), dimension \( k_0 + k_1 + k_2 \), and minimum distance \( \delta \).

<table>
<thead>
<tr>
<th>No.</th>
<th>( \phi(C) )</th>
<th>No.</th>
<th>( \phi(C) )</th>
<th>No.</th>
<th>( \phi(C) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>[17, 8, 4]</td>
<td>12</td>
<td>[39, 12, 8]</td>
<td>22</td>
<td>[87, 51, 8]</td>
</tr>
<tr>
<td>3</td>
<td>[19, 9, 4]</td>
<td>13</td>
<td>[41, 14, 8]</td>
<td>23</td>
<td>[91, 40, 12]</td>
</tr>
<tr>
<td>4</td>
<td>[27, 17, 4]</td>
<td>14</td>
<td>[43, 15, 8]</td>
<td>24</td>
<td>[93, 42, 12]</td>
</tr>
<tr>
<td>6</td>
<td>[47, 36, 4]</td>
<td>16</td>
<td>[59, 29, 8]</td>
<td>26</td>
<td>[105, 52, 12]</td>
</tr>
<tr>
<td>7</td>
<td>[75, 45, 7]</td>
<td>17</td>
<td>[61, 30, 8]</td>
<td>27</td>
<td>[113, 56, 12]</td>
</tr>
<tr>
<td>8</td>
<td>[77, 47, 7]</td>
<td>18</td>
<td>[63, 31, 8]</td>
<td>28</td>
<td>[115, 57, 12]</td>
</tr>
<tr>
<td>9</td>
<td>[79, 48, 7]</td>
<td>19</td>
<td>[65, 32, 8]</td>
<td>29</td>
<td>[103, 26, 18]</td>
</tr>
<tr>
<td>10</td>
<td>[95, 62, 7]</td>
<td>20</td>
<td>[67, 35, 8]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

collectively as DNA codes. The strands, i.e., DNA strings, are preferred to be short to make the synthesis easy and cheap. They must, however, satisfy numerous constraints to be useful for applications. The two most common applications are as basic tools for biomolecular computation and as biomolecular barcoding-tagging system to identify and manipulate individual molecules in complex libraries.

Numerous approaches to DNA codes have been extensively investigated. A recent addition to several surveys that have appeared in the literature is the work of Limbachiya et al. in [12]. Tools from algebraic coding theory, both from finite fields as well as rings, have been fruitfully used since the inception. A relatively early work by Marathe et al. in [13] discussed important design criteria and bounds derived from error-correcting codes. We continue on this line of studies by constructing \( F_4R \)-DNA skew cyclic codes.

The Watson-Crick complement of a strand is the strand obtained by replacing each A by T and vice versa, and each G by C and vice versa. One writes \( \overline{A} = T, \overline{T} = A, \overline{C} = G, \text{ and } \overline{G} = C \). Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \) be distinct codewords in a DNA code \( \mathcal{D} \). The reverse of \( \mathbf{x} \) is \( \mathbf{x}_{\text{rev}} = (x_n, x_{n-1}, \ldots, x_1) \). The complement of \( \mathbf{x} \) is \( \mathbf{x}^c = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}) \). Hence, \( \mathbf{x}_{\text{rev}}^c = (\overline{x_n}, \overline{x_{n-1}}, \ldots, \overline{x_1}) \) is the reverse complement of \( \mathbf{x} \).

The process in which a strand and its complement bound to form a double-helix is known as hybridization. Constraints on the codewords in a DNA code are imposed to avoid it. Let \( \mathcal{D} \) be a DNA code of fixed length \( n \), cardinality \( M \), and minimum distance \( d \). Then the constraints on the Hamming distances

\[
\text{wt}_H(\mathbf{x}, \mathbf{y}) \geq d \text{ and } \text{wt}_H(\mathbf{x}^c, \mathbf{y}_{\text{rev}}) \geq d \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{D}
\]

are imposed to prevent hybridization between any two strands as well as between a strand and the reverse of any other strand. A reverse-complement DNA code \( \mathcal{D} \) has parameters \( (n, M, d, 4) \) that satisfies Equation (11). It is known, for instance, that any \( F_4 \)-cyclic code with generator polynomial \( f(X) \) is reverse-complement if and
only if \( f(X) \in \mathbb{F}_4[X] \) is a self-reciprocal polynomial, i.e., \( f(X) = X^{\deg(f(X))} f(X^{-1}) \), not divisible by \( X - 1 \).

Abualrub et al. studied \( \mathbb{F}_4 \)-DNA codes of odd lengths in [2] where they use the bijection between the set of DNA alphabets \( \{ A, T, C, G \} \) and \( \mathbb{F}_4 := \{ 0, 1, w, w^2 \} \), in that respective ordering. We extend this idea by letting, for all \( a \in R \),

\[
\theta(a) + \theta(\overline{a}) = v + 1. \tag{12}
\]

**Lemma 6.1.** For all \( a, b \in R \), we have

(i) \( a + b = \overline{a} + \overline{b} + v \)

(ii) \( (v + 1) a = (v + 1) \overline{a} + v \).

(iii) \( \overline{\overline{a}} = \overline{a} \).

The map defines the following bijection between the elements of \( R \) and the 16 codons in \( \{ A, C, G, T \}^2 \).

<table>
<thead>
<tr>
<th>( a \in R )</th>
<th>Codon</th>
<th>( a \in R )</th>
<th>Codon</th>
<th>( a \in R )</th>
<th>Codon</th>
<th>( a \in R )</th>
<th>Codon</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>AA</td>
<td>( v + w )</td>
<td>GT</td>
<td>( v w^2 + 1 )</td>
<td>GG</td>
<td>( v w )</td>
<td>CC</td>
</tr>
<tr>
<td>( v )</td>
<td>TT</td>
<td>( w )</td>
<td>CA</td>
<td>( v w^2 + w^2 )</td>
<td>AG</td>
<td>( v w + w^2 )</td>
<td>TC</td>
</tr>
<tr>
<td>( v + 1 )</td>
<td>AT</td>
<td>( v + w^2 )</td>
<td>CT</td>
<td>( v w^2 )</td>
<td>GG</td>
<td>( v w + w )</td>
<td>AC</td>
</tr>
<tr>
<td>1</td>
<td>TA</td>
<td>( w^2 )</td>
<td>GA</td>
<td>( v w^2 + w )</td>
<td>TG</td>
<td>( v w + 1 )</td>
<td>GC</td>
</tr>
</tbody>
</table>

**Definition 6.2.** An \( R \)-linear code \( C \) of length \( \beta \) is called DNA-skew cyclic if

1. The code \( C \) is \( R \)-skew cyclic of length \( \beta \).
2. For any codeword \( x \in C \), \( x \neq x^\text{rev} \) with \( x^\text{rev} \in C \).

We adopt the following definition of reciprocal polynomials and a useful lemma from [5].

**Definition 6.3.** Let \( f(X) = f_0 + f_1 X + \ldots + f_k X^k \) be a polynomial in \( R[X, \theta] \). The reciprocal polynomial of \( f(X) \) is the polynomial \( f^*(X) \) given by

\[
f^*(X) = f_0 X^k + f_1 X^{k-1} + f_2 X^{k-2} + \ldots + f_{k-1} X + f_k. \tag{13}
\]

If \( f(X) = f(X)^* \), then \( f(X) \) is self-reciprocal.

**Lemma 6.4.** [5] Let \( f(X), g(X) \in R[X, \theta] \) with \( \deg(f(X)) \geq \deg(g(X)) \). Then the following assertions hold.

1. \( (f(X)g(X))^* = f(X)^* \cdot g(X)^* \)
2. \( (f(X) + g(X))^* = f(X)^* + g(X)^* X^{\deg(f(X)) - \deg(g(X))} \).

A code is reversible complement if \( x^\text{rev} \in C \) for any \( x \in C \). The next theorem characterizes reversible complement \( R \)-skew cyclic code.

**Theorem 6.5.** Let two polynomials \( g_1(X) \) and \( g_2(X) \) divide \( X^\beta - 1 \) in \( \mathbb{F}_4[X] \). Let \( C = \langle g(X) \rangle \) be \( R \)-skew cyclic with \( g(X) = v g_1(X) + (v + 1) g_2(X) \). Then \( C \) is reversible complement if and only if \( g(X) \) is self-reciprocal and \( v(X^\beta - 1)/(X - 1) \in C \).

**Proof.** Let \( g(X) = v g_1(X) + (v + 1) g_2(X) \) and \( C = \langle g(X) \rangle \) be an \( R \)-skew cyclic code of length \( \beta \). Suppose that \( C \) is reversible complement. Since \( 0 \in C \), we have \( (0, 0, \ldots, 0) = (v, v, \ldots, v) = v(X^\beta - 1)/(X - 1) \in C \). Let

\[
g_1(X) = g_0 + g_1 X + \ldots + g_{k-1} X^{k-1} + X^k \quad \text{and} \quad g_2(X) = h_0 + h_1 X + \ldots + h_{k-1} X^{k-1} + X^k
\]
where \( t \leq k \). Then
\[
g(X) = vg_1(X) + (v + 1)g_2(X)
\]
\[
= (vg_0 + (v + 1)h_0) + (vg_1 + (v + 1)h_1)X + \ldots
\]
\[
+ (vg_{t-1} + (v + 1)h_{t-1})X^{t-1} + (v + (v + 1)h_t)X^t
\]
\[
+ (v + 1)h_{t+1}X^{t+1} + \ldots + (v + 1)h_{k-1}X^{k-1} + (v + 1)X^k.
\]

Since \( C \) is reversible complement, it contains
\[
g_{rev}(X) = v(1 + X + \ldots + X^\beta-k-2) + (v+1)X^{\beta-k-1} + (v+1)h_{k-1}X^{\beta-k}
\]
\[
+ \ldots + (v+1)h_{t+1}X^{\beta-t-2} + (v + (v + 1)h_t)X^{\beta-t-1} + vX^{\beta-t-1}
\]
\[
+ (vg_{t-1} + (v + 1)h_{t-1})X^{\beta-t} + \ldots + (vg_1 + (v + 1)h_1)X^{\beta-2} + (v + 1)h_0X^{\beta-1}.
\]

Using Lemma 6.1 we can write
\[
g_{rev}(X) = v(X^{\beta - 1})/(X - 1) \in C. \text{ This implies }
\]
\[
g_{rev}(X) + v(X^{\beta - 1})/(X - 1) =
\]
\[
(\frac{(v + 1) + v}{(v + 1)h_{t+1}X^{\beta-1}} + (\frac{(v + 1) + v}{(v + 1)h_{k-1} + v})X^{\beta-k} + \ldots + \frac{(v + 1)h_{t+1}X^{\beta-t-2} + (v+v)X^{\beta-t-1} + (v+1)h_tX^{\beta-t-1} + vX^{\beta-t-1}}{v(v+1)h_1 + v}X^{\beta-t-1} + v(v+1)h_{t-1}X^{\beta-t} + vX^{\beta-t} + \ldots + v(v+1)h_1X^{\beta-2} + v(v+1)h_0X^{\beta-1} + (v+1)h_0X^{\beta-1})
\]

By Equation (12) we can write
\[
(v + 1)X^{\beta-k-1} + (v + 1)h_{k-1}X^{\beta-k} + \ldots + (v + 1)h_{t+1}X^{\beta-t-2} + vX^{\beta-t-1} + (v + 1)h_tX^{\beta-t} + vX^{\beta-t} + (v + 1)h_{t-1}X^{\beta-t} + \ldots + vX^{\beta-t} + (v + 1)h_1X^{\beta-2} + vX^{\beta-1} + (v + 1)h_0X^{\beta-1}
\]
as
\[
(v + 1)X^{\beta-k-1} + (v + 1)h_{k-1}X^{\beta-k} + \ldots + (v + 1)h_{t+1}X^{\beta-t-2} + (v + (v + 1)h_t)X^{\beta-t-1} + (vg_{t-1} + (v + 1)h_{t-1})X^{\beta-t} + \ldots + (vg_1 + (v + 1)h_1)X^{\beta-2} + (v + 1)h_0X^{\beta-1} + vX^{\beta-1}
\]

Multiplying on the right by \( X^{k+1-\beta} \), we obtain
\[
(v + 1) + (v + 1)h_{k-1}\theta(1)X + \ldots + (v + 1)h_{t+1}\theta(1)X^{k-t-1}
\]
\[
+ (v + 1)h_t\theta(1)X^{k-t} + (vg_{t-1} + (v + 1)h_{t-1})\theta(1)X^{k-t+1} + \ldots + (vg_1 + (v + 1)h_1)\theta(1)X^{k-1} + (v + 1)h_0\theta(1)X^k.
\]
Hence, \( g^*(X) \in C \). Since \( C = \langle g(X) \rangle \), there exists \( q(X) \in R[X, \theta] \) such that \( g^*(X) = q(X) \ast g(X) \), which implies \( \text{deg}(g^*(X)) = \text{deg}(g(X)) \) and \( q(X) = 1 \). Thus, \( g^*(X) = g(X) \), as required.

Conversely, let \( C = \langle g(X) \rangle \) be an \( R \)-skew cyclic code of length \( \beta \) generated by \( g(X) = v g_1(X) + (v + 1) g_2(X) \) where \( g_1(X) \) and \( g_2(X) \) are two divisors of \( X^\beta - 1 \) in \( \mathbb{F}_4[X] \). Let \( c(X) = c_0 + c_1 X + \ldots + c_k X^k \in C \), then there exist \( q(X) \in R[X, \theta] \) such that \( c(X) = q(X) \ast g(X) \). By Lemma 6.4, \( c^*(X) = q^*(X) \ast g^*(X) \). Since \( C \) is self reciprocal, \( c^*(X) = q^*(X) \ast g(X) \in C \) for any \( c(X) \in C \). Since \( C \) is skew cyclic, \( c(X) \ast X^{\beta-k-1} = c_0 X^{\beta-k-1} + c_1 X^{\beta-k} + \ldots + c_k X^{\beta-1} \in C \). Hence, \( v(X^\beta - 1)/(X - 1) = v(1 + \ldots + X^{\beta-1}) \in C \). Since \( C \) is \( R \)-linear,

\[
c(X) \ast X^{\beta-k-1} + v(X^{\beta-1})/(X - 1) = v + \ldots + vX^{\beta-k-2} + (c_0 + v)X^{\beta-k-1} + \ldots + (c_k + v)X^{\beta-1}.
\]

By Equation (12),

\[
v + \ldots + vX^{\beta-k-2} + v c_0 X^{\beta-k-1} + \ldots + v R X^{\beta-1} = (c^*(X))_{\text{rev}} \in C.
\]

This concludes the proof. \( \square \)

The theorem that we have just proved leads us from \( R \)-skew cyclic code to the definition and subsequent characterization of \( \mathbb{F}_4 R \)-skew cyclic code in the context of DNA coding.

**Definition 6.6.** An \( \mathbb{F}_4 \)-linear code \( C \) is DNA-skew cyclic if the followings hold.

1. \( C \) is an \( \mathbb{F}_4 R \)-skew cyclic code, i.e., \( C \) is an \( R \)-left submodule of \( F[X]/(X^\alpha - 1) \times R[X, \theta]/(X^\beta - 1) \).

2. Any codeword \( c = (c_1, c_2) \in C \) and its reverse complement \( c_{\text{rev}} = ((c_1)_{\text{rev}}, (c_2)_{\text{rev}}) \in C \)

must be distinct.

The characterization of reverse complement codes over \( \mathbb{F}_4 R \) can now be established.

**Theorem 6.7.** Let \( C = \langle (f(X), 0), (0, g(X)) \rangle \) be an \( \mathbb{F}_4 R \)-skew cyclic code. Note that \( \ell(X) := 0 \) and \( C = C_1 \otimes C_2 \) with \( C_1 \) an \( \mathbb{F}_4 \)-cyclic code and \( C_2 \) an \( R \)-skew cyclic code. Then \( C \) is reversible complement if and only if \( C_1 \) and \( C_2 \) are reversible complement over \( \mathbb{F}_4 \) and \( R \), respectively.

**Proof.** Let \( C \) be an \( \mathbb{F}_4 R \)-skew cyclic code generated by \( (f(X), 0) \) and \( (0, g(X)) \). Lemma 4.3 shows how to find \( C = C_1 \otimes C_2 \). Let \( c = (c_1, c_2) \in C = C_1 \otimes C_2 \) with \( c_1 \in C_1 \) and \( c_2 \in C_2 \). Suppose that \( C_1 \) and \( C_2 \) are reversible complement over \( \mathbb{F}_4 \) and \( R \), respectively. Then we have \( (c_1)_{\text{rev}} \in C_1 \) and \( (c_2)_{\text{rev}} \in C_2 \). Thus, \( ((c_1)_{\text{rev}}, (c_2)_{\text{rev}}) = c_{\text{rev}} \in C_1 \otimes C_2 = C \).

Conversely, let \( c_1 \in C_1 \) and \( c_2 \in C_2 \). Then \( c = (c_1, c_2) \in C \). If \( C \) is reversible complement, then \( c_{\text{rev}} = ((c_1)_{\text{rev}}, (c_2)_{\text{rev}}) \in C = C_1 \otimes C_2 \). This implies \( c_{1,\text{rev}} \in C_1 \) and \( c_{2,\text{rev}} \in C_2 \), as required. \( \square \)
7. Conclusion. We have presented our studies on skew cyclic codes over the ring $F_4R$. Their algebraic structures as left submodules of a skew-polynomial ring are investigated, resulting in the identification of their generators. The fact that, under some simple conditions on their length, they are equivalent to cyclic or 2-quasi-cyclic codes over the same ring is established. Towards the end we show how the setup leads naturally to DNA codes and prove a condition on the associated generator polynomial of an $F_4R$-skew cyclic code that guarantee the code to be reversible complement. We are now looking into whether the class of codes that we propose here contains those with better relative distance or size than known DNA codes.

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REFERENCES


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