Global Consensus Making on Multiplex Scale-Free Networks

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Abstract—Most individuals, if not all, are active in various social networks. Opinion formation is an outcome of social interactions and information propagation occurring in such social systems. In this paper, we study on opinion formation on multiplex networks composed of two layers, each of which is a scale-free network, with a special focus on global consensus making in such networks. We impose a new rule of pair-wise interactions under inter-layer interplay termed as compromise following which two interacting nodes have a tendency to make a fair compromise in both senses of inter- and intra-layer interplay. It is found that in a duplex network composed of two identical layers, an increase in tolerance range in a layer declines the opinion diversity on the other layer and the two critical confidence bounds for achieving global consensus in both layers follow a one-sum rule; that is, each of the layers reaches a global consensus if the sum of two critical bounds on the two layers is approximately equal to 1, a double of critical bound on a single-layer network. However, in duplex networks of two non-identical layers with layer-layer coupling quantified by a link overlap parameter, the rule does not hold any longer. In this case, due to the larger portion of low-degree nodes in typical scale-free networks, a layer only can reach a complete consensus if its associated tolerance range is as large as nearly 0.5 even when the tolerance range on the other layer is utmost.

Keywords—opinion dynamics; complex networks, scale-free networks

I. INTRODUCTION

Opinion formation is an interesting topic in the study of complex networks generally and computational sociology particularly [1, 2]. It plays an important role in predicting which opinions may dominate in a population after a temporal process of spreading through social interactions [3, 4]. Such studies reveal that local pair-wise interactions of individuals may eventually help form a global equilibrium.

Numerous opinion spreading models have been proposed to reveal system dynamics and evolution by introducing various rules of communication between individuals. Some of the most well-known ones include Voter model [5, 6], major rule model [7], Sznajd model [2], etc. Proposed by Deffuant et al. in 2000, the Deffuant model [8], as one of the most popular bounded confidence models, has grabbed much attention and been extensively studied over the past two decades. It is found that the tolerance bound (also interpreted as confidence bound, tolerance range, and uncertainty threshold etc. [9–11]) plays an important role in opinion formation processes. A notion of critical confidence bound is captured to refer to a threshold beyond which the population eventually converges to a single opinion cluster sharing the same opinion (termed as global consensus in this report) that equals the average of first impression [8, 12, 13]. Extensive simulations also reveal another interesting result that the number of coexisting opinion clusters in the final configuration is approximately 1/(2*tolerance range) [8, 14]. Some analytical frameworks and theoretical approaches have been developed using the mean-field approach [15–17]. Such an approach adopts a method in statistical physics by deriving a rate equation describing the system dynamics, and as such enables one to reproduce the whole evolution process. The model has also been modified by replacing continuous-valued opinion with discretized values [18].

Though the studies on opinion formation have achieved remarkable results, most of the existing works focus on isolated and non-interacting networks. Many questions hence remain open as many modern real-life systems networks, inclusive of social networks, are composed of layers coupled together, termed as multiplex networks [19–21]. In computational sociology, some frameworks have taken a first step towards discovering new properties in opinion evolution on multiplex networks. The authors of [22] developed an analytical framework examining the Deffuant model on a one-dimensional multiplex network. The work was rigorously conducted to draw an interesting conclusion that under a predefined configuration of multiplexity, the existence of interconnected layers impedes the convergence. However, it is not exhaustive to obtain a universal conclusion in a general case where network structures are diverse rather than ring-like layers. Another consensus model validated in allocation problems, in contrast, concluded that more dimensions lead to better chances for consensus [23], revealing the potential difficulties in drawing general conclusions within a certain general framework.

In this paper, we study the Deffuant model based opinion dynamics on two-layer multiplex networks. On each layer, the agents hold a type of opinion on a subject associated with the underlying layer. The regime of inter-layer interactions mimics
the compromise phenomenon in real-life social networks where one's viewpoint on a subject is influenced by his/her opinion on other subjects, resulting in their decisions on compromising or neglecting others' views. An investigation is also carried out to examine the impact of the strength of inter-layer coupling qualified by link overlap on opinion formation.

In this study, scale-free networks are of interest due to their ubiquity in natural and social systems and networks. Note that different network topologies characterized by degree distribution may lead to significant differences in the dynamics and global conclusions. Due to limited space, this paper only presents results observed in scale-free networks.

The rest of this paper is organized as follows. The opinion dynamic model on duplex network is introduced in Section II. Simulation results and discussions are presented in Section III. Section IV concludes the paper and briefly discusses on future research direction.

II. DUPLEX OPINION DYNAMICS MODEL

In the present model, we construct a duplex network composed of 2 layers. The duplex network is a pair $G = (V, E)$ consisting of 2 layers $G_1$ and $G_2$, each of which is a scale-free network $G_i = (V_i, E_i)$, where $V_i$ is the set of vertices representing individuals in a population and $E_i$ is the set of edges on layer $i$, $i = 1, 2$. The edge set of $G$, denoted by $E$, consists of two sets of edges implying two types of social relationships: $E = E_1 \cup E_2$. The artificial scale-free networks are constructed using the generative model UCM [24]. Given an average degree $\langle k \rangle$, the scaling exponent of the power-law function can be obtained by numerically solving the following equation:

$$\sum_k kp(k) = \langle k \rangle,$$

where $p(k) = \frac{k^{-\gamma}}{\zeta(\gamma, k_{\min}) - \zeta(\gamma, k_{\max})}$, $k_{\min}$ and $k_{\max}$ are the minimum and maximum degrees respectively, and $\zeta(\gamma, k)$ is the Hurwitz $\zeta$ function [25, 26].

The coupling between layers has significant effects on the universal opinion formation. We quantify this interlayer coupling by a link overlap parameter that is denoted as $\eta$ and obtained by computing the fraction of cardinalities $\frac{|E_1 \cap E_2|}{|E_1|}$.

In order to make a comparison focusing on effects of the overlap parameter value, we deliberately keep the degree of all nodes on both the layers to be the same. Specifically, given a network $G_1$, a scheme for constructing a layer $G_2$ that meets a desired $\eta$ is described as follows: in case of $\eta = 1$, the two layers are of exactly the same topology; otherwise, perform the following rewiring process: start with an initial $G_2 \equiv G_1$; randomly choose nodes $A, B, C, D \in V$ where links $\{A, B\}, \{C, D\} \in E_2$ and $\{A, C\}, \{B, D\} \in E_2$, then replace the links $\{A, B\}, \{C, D\}$ with $\{A, C\}$ and $\{B, D\}$, respectively; repeat such rewiring operations until the pre-specified $\eta$ is met. All nodes in the resulting layer $G_2$ have the same nodal degrees as their corresponding nodes in $G_1$.

To study on opinion formation in multiplex networks, we introduce a rule of pair-wise communication between individuals on each layer (termed as intralayer interaction [27]) under interlayer interplay. Firstly, we recall the rule of opinion exchange in a single network following the Deffuant model: initially, each node in the network is assigned a continuous-valued opinion drawn from a uniform distribution in the region $[0, 1]$. At time step $t$, if a randomly selected edge $e = \{u, v\}$ is active, the opinions held by $u$ and $v$ at time step $t + 1$ are determined by the following rule: if $|o(u, t) - o(v, t)| \leq d$, where $o(u, t)$ and $o(v, t)$ denote the opinions held by the node $u$ and $v$ at time $t$, respectively, and $d$ is the tolerance bound, then the two nodes make consensus:

$$o(u, t + 1) = o(u, t) + \mu(o(v, t) - o(u, t)),$$
$$o(v, t + 1) = o(v, t) + \mu(o(u, t) - o(v, t)),$$

where $\mu \in (0, 1/2]$ is convergence parameter; otherwise, they remain their opinions. According to most of the existing results, $\mu$ only has effect on convergence time of the opinion dynamics [8, 23, 28]. Therefore, we fix $\mu = 1/2$ in the remainder of this paper.

In the duplex network, on each of the constituent layers, say layer 1, at the beginning each node $v$ holds an opinion value that is drawn from a uniform distribution over the region $[0, 1]$, denoted by $o_s(v, 0)$. At every time step, each of the two layers is active with an equal probability of 0.5. On the active layer, randomly choose a node $u$ and then among its neighbors, pick a node $v$ at random. We introduce a communication rule termed as interlayer compromise to account for the interplay of the two layers. The rule is described as follows: if $u$ and $v$ have connection only on the active layer, e.g. layer 1, they behave the same as that in a single network. In other words, if they are not connected on the other layer, i.e., layer 2, their opinions on layer 2 do not interfere the opinion exchange process occurring on layer 1. Otherwise, they make consensus on layer 1 if and only if the following condition holds:

$$\frac{|o_1(u, t) - o_1(v, t)| + |o_2(u, t) - o_2(v, t)|}{2} \leq d_1 + d_2,$$

where $d_j, j \in \{1, 2\}$, is the confidence bound on layer $j$ and independent of each other. The opinion updating process proceeds until both the layers reach a quasi-steady state where on each layer, any pair of neighboring nodes either shares
approximately identical opinions or never makes consensus in any future contact according to the compromise rule as defined.

To evaluate the degree of consensus in the opinion systems, we employ the quantity of the number of opinion clusters co-existing at the equilibrium state. Such a number is calculated by firstly removing links connecting nodes that would never make consensus according to the compromise rule and then computing the number of disconnected components in the layers by using the Dulmage-Mendelsohn decomposition algorithm [29]. Note that in the present framework, the term global consensus is defined in the strictest sense, i.e., the whole population finally concurs and the number of surviving cluster, therefore, equals 1. This is important since typically, several isolated nodes and minor groups, even of size one, still exist in the final state, called stubborn agents, extremists or outliers in some senses [18, 30], and remain their opinions even if a giant fraction of the population already converges to a major opinion cluster. The corresponding critical confidence bound, hence, may vary according to criteria for global consensus.

In the present model, the component layers are supposed to play the same role, i.e. un-weighted layers. Therefore, the opinion spreading processes on both layers occur concurrently and similarly. For this reason, we mainly discuss the opinion dynamics on layer 1 under the effects of layer 2 in comparison to the case in single networks.

III. SIMULATION RESULTS AND DISCUSSIONS

To get a first insight, we plot the number of opinion clusters co-existing on layer 1 at the final steady state as a function of $d_1$ and $d_2$ in different cases of $\eta$. When $\eta = 0$, no pair of the nodes has connections on both layers. The cluster formation on the layer 1, therefore, is not influenced by $d_2$ as observed in Fig. 1(a). For $\eta = 1$, the two layers are fully coupled. As a result, $d_2$ has a significant effect on the layer 1’s opinion system. The number of clusters declines as either $d_1$ or $d_2$ increases but remains unchanged along the lines $d_1 + d_2 = $ constant. The phenomenon becomes much more complicated when $\eta = 0.5$. Here we proceed to get deeper insights by starting with a duplex network of identical layers.

Figure 2 provides a more detailed look on the diversity of opinion as $d_2$ changes. Similarly to the case of single networks, the number of opinion groups declines as $d_1$ increases. Obviously, the formation of opinion clusters on layer 1 is affected significantly by the presence of layer 2 characterized by various $d_2$. At a very small $d_2$, e.g., $d_2 = 0.01$, the number of surviving clusters saturates at the size of the network when $d_1$ approaches 0, similar to the phenomenon observed in single networks [10]. However, the fragmentation of the opinion system is severer. This can be understood: when $d_2$ approaches 0, two nodes on layer 1 can make local consensus if and only if $|\alpha_1(u,t) - \alpha_1(v,t)| + |\alpha_2(u,t) - \alpha_2(v,t)| \leq d_1$, meaning that the closeness in the nodes' opinions needs to be under a value less than its layer's tolerance bound $d_1$, thus making the individuals harder to make compromise in pair-wise communication sessions and finally diverse to different opinion groups.

In a duplex network of two identical layers with both tolerance ranges playing the same role, it is easy to understand that an increase in $d_2$ also decreases the diversity of opinion clusters on layer 1 in the similar way as $d_1$ does. Noticeably, the consensus-to-polarized transition point (as termed in [10]) shifts to the left along $d_1$ axis as $d_2$ increases and vanishes.

![Fig. 1. Number of coexisting opinion clusters on layer 1 at steady state as a function of tolerance ranges $d_1$ and $d_2$ in cases $\eta = 0$ (a), 0.5 (b), and 1 (c), respectively. The number of clusters is shown in color scale. Each layer is a scale-free network with a size of $N = 5000$, an average nodal degree of $\langle k \rangle = 10$, a minimum degree of 4, and a cutoff degree of 70. Dash lines are boundaries of critical regions in which only a single opinion cluster is formed. The results are obtained from a single simulation run.](image)

![Fig. 2. Number of clusters at steady state as a function of $d_1$ on duplex network with a size of 5000 and various average nodal degrees in three cases of $d_2$: 0.01 (diamonds), 0.2 (triangles) and 0.4 (circles). Plot for single network case (squares) is added for reference. The results are averaged over 10 independent realizations.](image)

![Fig. 3. Sizes of the biggest (solid lines) and the second biggest (dash lines) clusters in the final state on single network (black) and layer 1 of duplex network with $d_2 = 0.01$ (red), 0.2 (green) and 0.4 (blue). All networks have a size of 5000 and an average nodal degree of 10.](image)
when $d_2$ is sufficiently large, e.g., $d_2 \geq 0.4$. It is because under the strong influence of layer 2 with a large $d_2$, a giant cluster on layer 1 containing a large proportion of the population may be formed at small values of $d_1$ as shown in Fig. 3. This also causes the extinction of second largest cluster.

We here have a look at the impact of interlayer coupling by examining the critical tolerance range on layer 1 for various values of $\eta$. By adopting the same trial-and-error approach, we determine $d_1^c$ in different cases of link overlap. An interesting observation is shown in Fig. 5: for a range of $d_2$ less than 0.5, the increase in the overlap of the links in the layers enlarges the critical tolerance range in layer 1, meaning that the layer 1 requires a higher threshold to reach a global consensus. This can be understood that an increase in $\eta$ intensifies the impact of the interlayer interdependence on the opinion formation on layer 1. Therefore, with a $d_2$ less than a critical threshold to make an isolated network completely consensual, i.e., $d^c = 0.5$, a larger critical bound on layer 1 is needed as a mechanism to compensate the potential fragmentation on layer 2's opinion system with a small $d_2$. However, for $d_2 > 0.5$, $d_1^c$ no longer declines with an increase in $d_2$ and remains at around 0.5. It is found that generally there are a number of minor opinion groups insisting on their different opinions in the final state even when the giant cluster sharing the same opinion is already formed up. It is more likely to happen in scale-free networks where low-degree agents constitute a large portion of population. These agents have fewer communication channels and fewer chances to encounter assentors, and consequently finally form into minor clusters, inclusive of isolated agents. In this case, even when the tolerance range on the other layer is larger than 0.5, the layer in question needs to have a tolerance range as larger as 0.5 to gain a global consensus.

IV. CONCLUSIONS

In this paper, we studied opinion formation on a duplex network, with a main focus on the special case of global consensus making, to investigate opinion dynamics in multiplex social networks. It is found that in a multiplex network of two identical layers, following the defined compromise rule in consensus making, the sum of the critical confidence bounds on the two layers is equal to 1, which is a double of the critical bound in an isolated network.

The phenomenon, however, changes greatly when the two layers are not fully coupled. The layer-layer coupling becomes an impeding factor for the global consensus formation on a layer when the magnitude of tolerance range on the other layer is smaller than 0.5. The investigation of non fully-coupled duplex networks also reveals that even when the link overlap parameter is large, e.g., as large as 0.8, the critical tolerance bounds on the two layers no longer obey the one-sum rule. Specifically, when a network is greatly open-minded with a tolerance larger than 0.5, it still cannot complement consensus formation on the other layer, i.e., a layer can reach a global consensus state only if its own confidence bound is sufficiently large to be around or larger than 0.5 in duplex scale-free networks, regardless of how large its coupled layer's confidence bound is.

As aforementioned, due to length limit, we have omitted discussions on duplex networks with other network topologies, with the understanding that network topology has significant
effects on dynamics of opinion formation in multiplex networks. Further details will be presented in a separate report.

Other topics of future research interest may include opinion formation in multiplex networks with additional other actions including link rewiring [10] and opinion mutation [31-33], etc.

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REFERENCES


