Abstract—A discrete-time tracking differentiator (TD) based on a time criterion is presented. Its control law is derived by comparing the time that the initial state is driven to the switching curve or the origin with the sampling period. The advantages of this TD include faster signal-tracking, better noise filtering and derivative extraction. Simulation results show that this TD has smaller errors in signal-tracking and derivative extraction compared with the existing differentiators. Experiments are carried out on the gap sensor of the suspension control system in a magnetic-levitation (maglev) train to demonstrate that the proposed TD can obtain an effective velocity signal from the gap sensor when the accelerometer fails.

Index Terms—Tracking differentiator, time criterion, discrete time, gap sensor, maglev train.

I. INTRODUCTION

The differentiation of a given signal in real time is a well-known yet challenging problem in control engineering and theory [1], [2]. The proportional-integral-derivative (PID) control law developed in the last century still plays an essential role in modern control-engineering practice [3], [4]. However, derivative signals are prone to the noise pollution and derivative controls usually cannot be physically implemented, causing the PID to be degraded to PI control [5], [6]. For this reason, much effort has been devoted to the problem of designing differentiator, such as high-gain observer-based differentiator [7], linear time-derivative tracker [8], super-twisting second-order sliding-mode algorithm [9], robust exact differentiator [10], and finite time-convergent differentiator [11], [12], among others [13]-[15].

A noise-tolerant time-optimal system-based tracking differentiator (TD) was first proposed by J.Q. Han [16], [17]. The advantage of this TD is that it sets a weak condition on the stability of the systems to be constructed for TD and requires a weak condition on the input. In addition, it has advantageous smoothness compared with the obvious chattering problem encountered by sliding-mode-based differentiators [18]. Han used this TD as an important part of the emerging active disturbance rejection control (ADRC) [18]. He also presented a nonlinear PID control based on a TD [19]. Therein, the TD acted not only as the derivative extraction, but also as a transient profile that the output of a plant could reasonably follow to avoid the setpoint jump in PID. However, to achieve high accuracy of Han’s TD in signal-tracking and derivative extraction requests imposing strict constraints on the sampling period. Furthermore, the control law (denoted as $F_{han}$) of the TD is determined by comparing the location of the initial state with the isochronic region that is obtained through complicated nonlinear boundary transformation [17].

In this paper, we propose a discrete TD that has characteristics of faster signal-tacking, greater accuracy in derivative extraction and better noise filtering. In addition, the control law is obtained easily by comparing the time that the initial state sequence is driven to the switching curve or the origin with the sampling period.

The remainder of the paper is organized as follows: in Section II, the control law of the TD via a time criterion is presented in detail. In Section III, the structure of the TD and its filtering characteristic are discussed. In Section IV, numerical simulation results are presented to compare the performance of signal-tracking and derivative extraction for the proposed TD versus the existing ones. Experiments are carried out to acquire effective velocity signals from the gap sensor for the suspension system in the maglev train when the accelerometer fails. In Section V, the conclusions and remarks are given.

II. CONTROL LAW FOR TRACKING DIFFERENTIATOR

Consider a standard discrete-time double-integral system as follows

$$x(k+1) = Ax(k) + Bu(k), |u(k)| \leq 1 \quad (1)$$

where

$$A = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ h \end{bmatrix}.$$

The goal is to derive a high-precision feedback-control law (denoted as $F_{td}$) based on a time criterion in discrete time domain, which is described as follows:

Control Objective: Given the system (1) and its initial state $x(0)$, determine the control signal sequence, $u(0), u(1), \ldots, u(k)$ by comparing the time that the initial state point is driven back to the switching curve or the origin with the sampling period $h$ such that the state $x(k)$ is driven back to the origin in a finite number of steps, subject to the constraint of $|u(k)| \leq 1$.

The time that any initial state point $M(x_1(0), x_2(0))$ is driven back to the switching curve ($\Gamma(x_1, x_2) = x_1 + \frac{x_2}{2}$) [20],
(21)) is denoted as \( t_A \), and the time that the state point located on the switching curve is driven back to the origin is denoted as \( t_B \). We can then determine that \( t_A = s x_2(0) + \sqrt{x_1(0) + \frac{1}{4}x_2(0)^2} \) and \( t_B = |x_2(0)| \), where \( s = -\text{sign}(x_1(0) + \frac{1}{4}x_2(0)) \). The work of determining the control signal sequence that drives any initial state to the origin can be divided into the following two tasks:

**Task 1**: Determine the control signal sequence when the initial state point \( M(x_1(0), x_2(0)) \) is not located on the switching curve by comparing the time \( t_A \) and the sampling period \( h \).

**Task 2**: Determine the control signal sequence when the state point \( M(x_1(0), x_2(0)) \) is located on the switching curve by comparing the time \( t_B \) and the sampling period \( h \).

For **Task 1**, when \( h \leq t_A \), the control signal is \( u = -s \); otherwise, the control signal value should be decreased to guarantee that the state point \( M(x_1(0), x_2(0)) \) can be driven to the switching curve \( \Gamma \) within the sampling time of \( h \). When the state point is located above the \( \Gamma \), there exists \( s = 1 \) and the control signal takes on \( u = -u_a \). \( u_a \) satisfies the following equations:

\[
\begin{align*}
    x_1 &= x_1(0) - \frac{1}{2u_a}(x_2^2 - x_2^2(0)) \\
    x_2 &= x_2(0) - u_a h
\end{align*}
\]  

(2)

There exists \( x_1 = \frac{1}{2}x_2^2 \) when the initial state point is driven back to the switch curve by the corresponding control signal sequence. If \( u_a \) is taken as an unknown, then

\[
\frac{1}{2}h^2 u_a^2 + \left( \frac{1}{2}h^2 - hx_2(0) \right) u_a + \frac{1}{2}x_2^2(0) - x_2(0)h - x_1(0) = 0.
\]

(3)

The discriminant of (3) is

\[
\Delta = \left( \frac{1}{2}h^2 - hx_2(0) \right)^2 - 2h^2 \left( \frac{1}{2}x_2^2(0) - x_2(0)h - x_1(0) \right)
\]

\[
= \frac{1}{4}h^4 + h^3 x_2(0) + 2h^2 x_1(0).
\]

Two cases are possible.

1) When \( x_2(0) \geq 0 \), the discriminant satisfies the condition that

\[
\Delta > \frac{1}{4}h^4 + 2h^2 (x_1(0) + \frac{1}{2}x_2^2(0))
\]

\[
= \frac{1}{4}h^4 + 2h^2 (x_1(0) + \frac{1}{2}x_2(0)|x_2(0)|) > \frac{1}{4}h^4 > 0;
\]

2) When \( x_2(0) < 0 \), \( x_1(0) - \frac{1}{2}x_2^2(0) > 0 \) can be derived because \( x_1(0) + \frac{1}{2}x_2(0)|x_2(0)| > 0 \). Therefore,

\[
\Delta = \frac{1}{4}h^4 + h^3 x_2(0) + 2h^2 x_1(0)
\]

\[
> \frac{1}{4}h^4 + h^3 x_2(0) + 2h^2 \frac{1}{2}x_2^2(0) = h^2 (x_2(0) + \frac{1}{2}h)^2 \geq 0.
\]

According to two cases analysed above, the discriminant can always satisfy \( \Delta > 0 \). Furthermore, there exists two unequal real-number roots that satisfy \( x_2(h) < 0 \). Because \( x_2(h) = x_2(0) - u_a h = \frac{1}{2} \pm \frac{\sqrt{\Delta}}{h} < 0 \), the positive root shall be excluded and the expression of \( u_a \) can be obtained as follows:

\[
u_a = -\frac{1}{2} + \frac{1}{h} x_2(0) + \frac{1}{2} \sqrt{1 + \frac{4}{h^2} x_2(0) + \frac{8}{h^2} x_1(0)}.
\]  

(4)

Similarly, when the state point \( M \) is located below \( \Gamma \) where \( s = -1 \), the control input is \( u = u_a \). \( u_a \) can be derived as follows:

\[
u_a = -\frac{1}{2} - \frac{1}{h} x_2(0) + \frac{1}{2} \sqrt{1 - \frac{4}{h^2} x_2(0) - \frac{8}{h^2} x_1(0)}.
\]  

(5)

The resulting expression of \( u_a \) is

\[
u_a = -\frac{1}{2} - \frac{s}{h} x_2(0) + \frac{1}{2} \sqrt{1 + \frac{4}{h^2} x_2(0) - \frac{8}{h^2} x_1(0)} s
\]  

(6)

where the parameter \( s \) has the same definition as described in the previous section.

For **Task 2**, when the state point \( M \) is located on the switching curve \( \Gamma \), it satisfies the condition

\[
\begin{align*}
    x_1(t) &= \frac{1}{2}x_2^2(t) \\
    x_2(t) &= x_2(0) + u t = 0, t = h.
\end{align*}
\]  

(7)

When \( h \leq t_B \) (\( t_B = |x_2(0)| \)), the control law is \( u = -\text{sign}(x_2(0)) \). However, when \( t > t_B \), the control signal value should be decreased to guarantee that the state point \( M \) can be driven back to the origin in one step. Therefore, the control signal cannot be the constant value but must satisfy the following condition:

\[
\begin{align*}
    x_1(t) &= x_1(0) + x_2(0)t + \int_0^t u(\tau) d\tau = 0 \\
    x_2(t) &= x_2(0) + \int_0^t u(\tau) d\tau = 0, t = h.
\end{align*}
\]  

(8)

For simplicity, supposing \( u = a + bt \), and substituting it into (8), we have

\[
\begin{align*}
x_2(0) + ah + \frac{1}{2}bh^2 &= 0 \\
x_1(0) + x_2(0)h + \frac{1}{2}ah^2 + \frac{1}{6}bh^3 &= 0.
\end{align*}
\]  

(9)

This leads to

\[
\begin{align*}
a &= -\frac{2}{h^2} (2x_2(0)h + 3x_1(0)) \\
b &= \frac{6}{h^3} (x_2(0)h + 2x_1(0)).
\end{align*}
\]  

(10)

From the above, we can obtain the control law based on a time criterion for a standard discrete-time double-integral system (1). This is denoted here as \( u(k) = Ftd(x_1(k), x_2(k), r, h) \), where the parameters \( r \) and \( h \) have the same definitions as described in the previous section. The resulting control law is

\[
\begin{align*}
u(k) &= Ftd(x_1(k), x_2(k), r, h) \\
x_1(k+1) &= x_1(k) + x_2(k) + \frac{1}{2}u(k) h^2 - \frac{1}{2}bh^3 \\
x_2(k+1) &= x_2(k) + u(k) h - \frac{1}{2}bh^3.
\end{align*}
\]  

(11)
For a general discrete time double-integral system, one can obtain its control law $Ftd$ by substituting the $x_1$ and $x_2$ with $z_1$ and $z_2$ respectively, where $z_1 = \frac{1}{c_1}$ and $z_2 = \frac{1}{c_2}$.

Based on the control law above, the following TD can be constructed:

$$ u(k) = Ftd(x_1(k) - v(k), c_1, x_2(k), r_0, c_0 h) $$

$$ x_1(k + 1) = x_1(k) + h x_2(k) + \frac{1}{2} u(k) h^2 - \frac{1}{2} b h^3 $$

$$ x_2(k + 1) = x_2(k) + u(k) h - \frac{1}{2} b h^3 $$

where $r_0$ is the quickness factor, $c_1$ is the damping factor, $c_0$ is the filtering factor, and $v$ is the given signal.

III. Structure Analysis of Discrete Time Tracking Differentiator

For a given signal sequence $v(k)(k = 0, 1, 2, ...)$, the discrete-time TD in (12) can be transformed into an approximately linear one by taking on the proper parameter $r$, as follows:

$$ x(k + 1) = G x(k) + H v(k), k = 0, 1, 2, ... $$

where $x = [x_1(k), x_2(k)]^T$, $H = [\frac{1}{c_1}, \frac{2}{c_0 c_1}]^T$ and $G = [G_1, G_2]^T$, where $G_1 = [1 - \frac{1}{c_1}, (1 - c_1) h]$ and $G_2 = [- \frac{2}{c_0} h, 1 - \frac{2 c_1}{c_0}]$.

We assume that the given signal is $v(t) = \sum_{i=1}^N A_i e^{j w_i t + \phi_i} + \xi(t)$, where $A_i, w_i \in \mathbb{R}^+$, $\phi_i \in \mathbb{R}$, and $\xi(t)$ is a width-steady process. Then we have

$$ x(k) = G^k p_0 + \sum_{i=1}^N (e^{j w_i h} f - G)^{-1} H A_i e^{j w_i (k h + \phi_i)} + \eta(k), $$

where $\eta(k + 1) = G \eta(k) + H \xi(k), k = 0, 1, 2, ...$, and $p_0$ is determined by the initial conditions $x(0)$ and $\eta(0)$. The sufficient and necessary condition for convergence of (14) is that the spectral radius of matrix $G$ satisfies $\rho(G) < 1$.

It can be derived that $x_1(k) = C G^k p_0 + \sum_{i=1}^N C (e^{j w_i h} f - G)^{-1} H A_i e^{j w_i (k h + \phi_i)} + C \eta(k)$ by choosing $C = [1, 0]$. When the transfer function of the discrete-time system is denoted as $F(z) = C(z I - G)^{-1} H$, $x_1(k)$ can be expressed as follows

$$ x_1(k) = C G^k p_0 + \sum_{i=1}^N \Phi(e^{j w_i h}) A_i e^{j w_i (k h + \phi_i)} + C \eta(k). $$

The characteristics of magnitude-frequency and phase-frequency for the TD in (12) are analysed in the followings. Given a sine signal $v(t) = A e^{j (\omega t + \phi)}$, the tracking output signal $y_{out}$ is also a sine signal when the particular signal sequence is long enough and the spectral radius of matrix $G$ satisfies $\rho(G) < 1$. We suppose that the output signal is $y_{out} = \beta v(t - \tau_0)$, where $\beta$ is the dynamic amplifier factor and $\tau_0$ is the time delay. By choosing the filtering factor $c_0$ to satisfy the condition $c_0 h \ll 1$, we have

$$ \beta = \frac{1}{1 + 0.5 c_0 (\tau_0 - 1) h - h^2} $$

$$ \tau_0 = (c_0 c_1 - 1) h $$

Furthermore, when the damping factor $c_1 = 1$, we have $\beta = 1$ and $\tau_0 = (c_0 - 1) h$. Based on the conditions above, the characteristics of magnitude-frequency and phase-frequency are shown in Fig. 1. We can see that the proposed tracking differentiator has good filtering ability when the filtering factor $c_0$ is properly selected.

![Fig. 1. Characteristic curves of magnitude-frequency and phase-frequency for TD.](image)

The above equation is a Lyapunov function of a discrete-time system, demonstrating the relationship between the output sequence’s variance $R$ and the filtering factor $c_0$. For the $Fhan$ algorithm, the matrices $G$ and $H$ are

$$ G = \left( \begin{array}{cc} 1 & h \\ \frac{1}{c_0 h} & 1 - \frac{2}{c_0} \end{array} \right), H = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) $$

respectively. By assuming that the density of the white noise power spectrum is $Q = 1$, the result shown in Fig. 2 is achieved.

As shown in Fig. 2, the tracking differentiator can filter random noises when the proper filtering factor $c_0$ is selected. The filtering capacity is better for the $Ftd$ algorithm than for the $Fhan$ algorithm.
IV. NUMERICAL SIMULATION AND EXPERIMENT RESULTS

The numerical simulations are conducted to compare the proposed differentiator versus the existing ones in signal-tracking and differentiation acquisition. Experiments are also carried out on the gap sensor of the suspension system of a maglev train to determine whether the proposed TD can acquire an effective velocity signal from the gap sensor when an accelerometer fails.

A. Numerical Simulation

In this subsection, we present some numerical simulation results to compare the following three differentiators.

DI. Robust exact differentiator using sliding mode technique from [10].
\[
\begin{align*}
\dot{x}_1 &= x_2 - \alpha |x_1 - v|^{0.5} \text{sign}(x_1 - v), \\
\dot{x}_2 &= -\beta \text{sign}(x_1 - v)
\end{align*}
\]

DII. Tracking differentiator based on discrete time optimal control $F_{\text{Fan}}$ from [16], [17].
\[
\begin{align*}
u(k) &= F_{\text{Fan}}(x_1(k) - v(k), x_2(k), r_0, c_0 h), \\
x_1(k + 1) &= x_1(k) + h x_2(k), \\
x_2(k + 1) &= x_2(k) + h u(k), |u(k)| \leq r
\end{align*}
\]

DIII. Tracking differentiator based on discrete time optimal control $F_{\text{td}}$ proposed in this letter.
\[
\begin{align*}
u(k) &= F_{\text{td}}(x_1(k) - v(k), c_1 x_2(k), r_0, c_0 h), \\
x_1(k + 1) &= x_1(k) + h x_2(k) + \frac{1}{2} u(k) h^2 - \frac{1}{2} b h^3, \\
x_2(k + 1) &= x_2(k) + u(k) h - \frac{1}{2} b h^3
\end{align*}
\]

The Matlab program of Euler method is adopted in investigation. We choose the same initial value $(x_1(0) = 0, x_2(0) = 2)$ and the input signal sequence $v(t) = 1 + \sin(3t) + \gamma(t)$ in all simulations, where $\gamma(t)$ is the evenly distributed white noise with an intensity of 0.001. For differentiator DI, the parameters are $\alpha = 1.5, \beta = 36$; for differentiator DII and differentiator DIII, the sampling step is $h = 0.005$, the quickness factor is $r_0 = 200$, the damping factor is $c_1 = 2$, and the filtering factor is $c_0 = 5$. As mentioned in proposition 2.1, the control signal in differentiator DIII satisfies $|u(k)| \leq r$. The comparison results in signal-tracking and differentiation acquisition for three differentiators are plotted in Figs. 3 and 4.

From above simulation results, we see that tracking differentiator DII and differentiator DIII based on discrete time optimal control are smoother than differentiator DI in which the discontinuous function produces chattering problem. From the perspectives of static errors and phase delays, differentiator DIII performs better in signal tracking and differentiation acquisition than differentiator DII.

B. Experiment Results

The experiments are intended to evaluate the practical engineering application of the proposed tracking differentiator during the operation of a maglev train. The experimental platform is shown in Fig. 5, where the maglev train adopts the real-time development environment RTW toolbox supported by Matlab software. The integrated electromechanical system of this train comprises the vehicle structure, the bogie, the track, and the suspension system [24]-[26]. The three electromagnets of the suspension system are regulated by three sets of controllers. The platform can carry out the single-point or multi-point modularity suspension control experiments because three suspension points are mechanically decoupled.

The suspension sensor group collects the gap, acceleration, and current signals from the suspension system. This closed-loop feedback-control scheme utilizes the PID control law [27], for which the velocity signal comes from the integral of the acceleration signal. Without that velocity signal, the suspension system cannot guarantee suspension stability. However, in a practical engineering scenario, the
accelerometer is more likely to fail because of poor operating conditions. Furthermore, redundant technology is not used for the accelerometer because it is too expensive. In our experiments, the proposed TD is proposed to track the gap signal and produce the velocity signal if an accelerometer fails.

The A/D module is set with a sampling frequency of $F_s = 2kHz$ to collect the gap signals and acceleration signals when the train is moving with load fluctuation and train body might not be stabilized at $3mm$. The proposed TD is used to track the gap signals and produce the corresponding velocity signals, where the filtering factor is $c_0 = 8$, the damping factor is $c_1 = 2$ and the quickness factor is $r = 650$. The comparative results are presented in Fig. 6. As shown in Fig. 6, the proposed TD is able to track the gap signals quickly with small tracking errors. It also produces the desired velocity signals with only a small phase delay. Therefore, if the accelerometer fails, we believe that the proposed TD can acquire the velocity signals effectively with the gap signals.

V. CONCLUSIONS

We proposed a new and simple control law that is effective for applications in discrete-time, double-integral systems. This algorithm enabled us to develop a novel tracking differentiator. Numerical simulation results demonstrated that, compared with the existing differentiators, the proposed noise-tolerant TD has faster signal-tracking, and better noise filtering and derivative extraction. Experiment results showed that the proposed noise-tolerant TD can produce effective velocity signals for the suspension system of a maglev train when the accelerometer fails. Note that the utility of the TD is not limited to signal-tracking and differentiation acquisition, it can also be adopted in constructing controllers. Future work will include analysing the accuracy of this tracking differentiator and further exploring its practical engineering applications.

ACKNOWLEDGEMENTS

This study is an outcome of the Future Resilient System (FRS) project at the Singapore-ETH Centre (SEC), which is funded by the National Research Foundation of Singapore (NRF) under its Campus for Research Excellence and Technological Enterprise (CREATE) program. Part of this work is also supported by the Ministry of Education (MOE), Singapore (Contract No. MOE 2016-T2-1-119) and Interdisciplinary Graduate School, Nanyang Technological University, Singapore.

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