

MAS436 Topology: Nets, Boole-Smith convergence, and Tychonoff Theorem

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1 Nets and Boole-Smith convergence

The material here is largely taken from the corresponding sections of [1]. I added more explanation here and there.

Directed sets are generalisations of sequences:

Definition 1. A poset D is called a directed set if for any $x, y \in D$ there exists a $z \in P$ such that $z \geq x, z \geq y$.

Example 2. In a topological space X , consider the set P of open sets containing a given point x . Define the partial order on P by inclusion: $U \geq V$ iff $U \subset V$. It is easy to see that P is a directed set: indeed, for any $U, V \in P$ also $U \cap V$ is in P , and $U \cap V \geq U, U \cap V \geq V$.

Definition 3. A net in a t.s. X is a function $\Phi : D \rightarrow X$, with D a directed set.

Definition 4. For $A \subset X$, a net Φ is said to be frequently in A if for any $b \in D$ there exists $D \ni c \geq b$ such that $\Phi(c) \in A$.

Φ is said to be eventually in A if there exists $b \in D$ such that $\Phi(c) \in A$ for all $c \geq b$.

Definition 5. Φ converges to $x \in X$ if Φ is eventually in U for any neighbourhood U of x . Such x is called a limit of Φ . Notation, when such x is unique: $x = \lim_{a \in D} \Phi(a)$.

Proposition 6. X is Hausdorff iff any two limits of a net in X are equal.

Proof. If Φ is eventually in $U, V \subset X$ then Φ is eventually in $U \cap V$. So if $U \ni x, V \ni y$ are open sets separating $x \neq y \in X$ then Φ cannot be eventually in U and V . Thus Φ has at most one limit in a Hausdorff space.

For the opposite direction, let $x \neq y \in X$ inseparable by open sets in X . Define $D = \{(U, V) \mid x \in U, y \in V\}$, with $(A, B) \geq (U, V)$ if $A \subset U, B \subset V$, and

$$\begin{aligned} \Phi : D &\rightarrow X \\ (U, V) &\mapsto z \in U \cap V. \end{aligned}$$

Then Φ converges to x and to y . Indeed, let $U \ni x$ and $V \ni y$ be open. Then any $(A, B) \geq (U, V)$ satisfies $\Phi((A, B)) \in U$, so Φ is eventually in U . Similarly, it is eventually in V . \square

Continuity and nets

Continuity can be described in terms of net convergence.

Proposition 7. *$f : X \rightarrow Y$ is continuous iff for every net $\Phi : D \rightarrow X$ converging to $x \in X$ also $f \circ \Phi : D \rightarrow Y$ converges to $f(x)$.*

Proof. Let f be continuous at x and $\Phi : D \rightarrow X$ a net converging to x . Then $U = f^{-1}(V)$, for an open $V \ni f(x)$, is open $x \in U$. Thus Φ is eventually in U . Hence $f \circ \Phi$ is eventually in V , as claimed.

Suppose now that f is not continuous. Then there exists open $V \subset Y$ such that $K = f^{-1}(V)$ is not open. Let $x \in K \setminus \text{int } K$, and D the directed set of open neighbourhoods U of x , as in Example 2. Define

$$\begin{aligned} \Phi : D &\rightarrow X \\ U &\mapsto z \in U \setminus K. \end{aligned}$$

If $B \ni x$ is open then $\Phi(U) \in B$ for any $U \geq B$ (recall that this means $U \subset B$), so Φ is eventually in B and thus converges to x . On the other hand $f(\Phi(U)) \notin V$, so $f \circ \Phi$ is not eventually in V . \square

A characterisation of the closure

In the case of metric space X , the closure \bar{A} of $A \subset X$ can be described in terms of limits of convergent sequences in A .

Proposition 8. *Given $A \subset X$, a subset of points in a topological space X , the set of limits of converging in X nets in A equals \bar{A} .*

Proof. Let $x \in \bar{A}$. Then any open $U \ni x$ satisfies $A \cap U \neq \emptyset$. Construct a net

$$\begin{aligned} \Phi : \{U \ni x\} &\rightarrow A \\ U &\mapsto x_U \in U \cap A. \end{aligned}$$

Obviously, Φ converges to x .

On the other hand, let $\Phi : D \rightarrow A$ be a net converging to x . By definition of convergence, Φ is eventually in any neighbourhood $U \ni x$. This any neighbourhood of x contains a point in A , implying $x \in \bar{A}$. \square

Subnets

The following definitions are inspired by the idea of (convergent) subsequences.

Definition 9. *A function $h : D' \rightarrow D$ is called final if for any $\delta \in D$ there exists $\delta' \in D'$ such that $\alpha' \geq \delta'$ implies $h(\alpha') \geq \delta$.*

Definition 10. A subnet of $\Phi : D \rightarrow X$ is the composition $\Phi \circ h$ of Φ with a final function $h : D' \rightarrow D$.

Proposition 11. $\Phi : D \rightarrow X$ is frequently in each neighbourhood of $x \in X$ iff Φ has a subnet that converges to x .

Proof. Consider a net $\mu : D \rightarrow X$ frequently in each neighbourhood $U \ni x$. Consider a directed set $D' = \{(\alpha, U) \mid \alpha \in D, x, \mu(\alpha) \in U\}$, so that the ordering is induced from \leq_D and inclusion of neighbourhoods of x , i.e. if $\alpha \leq \beta$ and $V \subset U$ then $(\alpha, U) \leq (\beta, V)$, for any $(\alpha, U), (\beta, V) \in D'$. Indeed, if $(\alpha, U), (\beta, V) \in D'$ then, as μ is frequently in $U \cap V$, there exists $\gamma \geq \alpha, \beta$ s.t. $\mu(\gamma) \in U \cap V$. Thus $(\gamma, U \cap V) \in D'$ and $(\gamma, U \cap V) \geq (\alpha, U), (\beta, V)$, so D' is directed, as claimed. We claim next that

$$\begin{aligned} h : D' &\rightarrow D \\ (\alpha, U) &\mapsto \alpha \end{aligned}$$

is final. Indeed, for any $\delta \in D$ one has $(\delta, X) \in D'$, and $(\alpha, U) \geq (\delta, X)$ implies $\alpha = h((\alpha, U)) \geq \delta$, as required.

Hence $\mu \circ h : D' \rightarrow X$ is a subnet of D . We claim that it converges to x . Let $N \ni x$ be a neighbourhood. As μ is frequently in N , by assumption, there is $\beta \in D$ s.t. $\mu(\beta) \in N$. If $(\alpha, U) \geq (\beta, N)$ then $\mu(h((\alpha, U))) = \mu(\alpha) \in U \subset N$. Thus $\mu \circ h$ is eventually in N .

The opposite statement is immediate. \square

Definition 12. A net $\Phi : D \rightarrow X$ is called universal if for any $A \subset X$ it is either eventually in A or eventually in $X \setminus A$.

Lemma 13. $f \circ \Phi$ is universal in Y for any universal $\Phi : D \rightarrow X$ and any function $f : X \rightarrow Y$.

Proof. Let $A \subset Y$. As Φ is universal, it is eventually either in $f^{-1}(A)$ or in $X \setminus f^{-1}(A)$. But $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$. Thus $f \circ \Phi$ is eventually either in A or in $Y \setminus A$. \square

The following is quite a surprise. We shall be using a form of the Axiom of Choice (so-called Zorn Lemma¹) in its proof.

Theorem 14. Every net $\Phi : P \rightarrow X$ has a universal subnet.

Proof. Consider all collections \mathbf{C} of subsets of X satisfying

$$A \in \mathbf{C} \Rightarrow \Phi \text{ is frequently in } A \tag{1}$$

$$A, B \in \mathbf{C} \Rightarrow A \cap B \in \mathbf{C}. \tag{2}$$

E.g. $\mathbf{C} = \{X\}$ is one such collection. Order these collections by inclusion. The union of any chain in this poset is certainly also one of these collections

¹If every chain in a poset has an upper bound then the poset contains a maximal element.

satisfying (1) and (2). By the Zorn Lemma, there will be a maximal by inclusion such a collection \mathbf{C}_0 .

Let $P_0 = \{(A, \alpha) \in \mathbf{C}_0 \times P \mid \Phi(\alpha) \in A\}$. The partial order

$$(B, \beta) \geq (A, \alpha) \Leftrightarrow B \subset A \wedge \beta \geq \alpha$$

makes P_0 into a directed set. The following function

$$\begin{aligned} h : P_0 &\rightarrow P \\ (A, \alpha) &\mapsto \alpha. \end{aligned}$$

is final. Thus we obtain a subnet $\mu = \Phi \circ h : P_0 \rightarrow X$. We claim that it is universal. Let $S \subset X$ be s.t. μ is frequently in S , i.e. for any $(A, \alpha) \in P_0$ there is $P_0 \ni (B, \beta) \geq (A, \alpha)$ s.t. $\mu((B, \beta)) = \Phi(\beta) \in S$. In particular, $B \subset A$ and $B \ni \beta \geq \alpha$. Thus $\Phi(\beta) \in S \cap B \subset S \cap A$. Therefore Φ is frequently in $S \cap A$ for any $A \in \mathbf{C}_0$.

Then we can add S and all the $S \cap A$ to \mathbf{C}_0 , and (1) and (2) would still hold. By maximality of \mathbf{C}_0 , we must have S there. If μ were also frequently in $X \setminus S$, then $X \setminus S \in \mathbf{C}_0$ would also hold, and by (2) also $\emptyset = S \cap X \setminus S \in \mathbf{C}_0$, violating (1). Thus μ is not frequently in $X \setminus S$. Therefore it is eventually in S .

We have shown that if μ is frequently in S then it is eventually in S . As for any $S \subset X$ and any net, it is frequently either in S or in $X \setminus S$, the claim that μ is universal follows. \square

2 Compactness, product topology, and nets

The main objective here is to derive Tychonoff's Theorem.

Proposition 15. *X is compact iff every net in X has a convergent subnet.*

Proof. To prove that compactness of X implies that every net has a convergent subnet, it suffices to show, in view of Theorem 14, that every universal net in X converges.

Suppose to the contrary that a universal net $\mu : D \rightarrow X$ does not converge, that is, for any $x \in X$ there is a open neighbourhood U_x such that μ is not eventually in U_x . As μ is universal, it must be eventually in $X \setminus U_x$, i.e., there exists $\beta_x \in D$ s.t. $\alpha \geq \beta_x$ implies that $x \notin U_x$. Let $\{U_{x_1}, \dots, U_{x_n}\}$ be a finite subcover of the cover $\{U_x \mid x \in X\}$, that must exist by compactness of X . Then $\mu(\alpha) \notin U_{x_i}$ as soon as $\alpha \geq \beta_{x_i}$, a nonsense, as $\mu(\alpha) \in X$. Thus every universal net in X converges.

To show the opposite, we show that the assumption that every net in X has a convergent subnet implies that every collection of closed sets in X with finite intersection property (f.i.p.) has a nonempty intersection.² Let \mathcal{C} be a

²f.i.p. for $\mathcal{A} = \{A_\omega\}$ means that every finite subset $\{A_1, \dots, A_k\}$ of \mathcal{A} satisfies $A_1 \cap \dots \cap A_k \neq \emptyset$. It is an exercise to show that compactness is equivalent to the property that every collection of closed sets with f.i.p. has a nonempty intersection.

collection of closed sets in X satisfying f.i.p. We can and will add all the finite intersections to \mathcal{C} without violating f.i.p. Then the inclusion makes \mathcal{C} into a directed set: $C \geq C'$ iff $C \subset C'$. Moreover, the following is a net in X :

$$\begin{aligned} \mu : \mathcal{C} &\rightarrow X \\ C &\mapsto x_C \in C. \end{aligned}$$

By assumption there must be a convergent to an $x \in X$ subnet $\mu \circ f$ given by a final map $f : D \rightarrow \mathcal{C}$. Thus for any $\alpha \in D$ we have $f(\alpha) \in \mathcal{C}$ and $\mu(f(\alpha)) = x_{f(\alpha)} \in f(\alpha)$. Let $C \in \mathcal{C}$. Then there is, by finality, $\beta \in D$ such that $D \ni \alpha \geq \beta$ implies $f(\alpha) \subset C$, and so $\mu(f(\alpha)) \in f(\alpha) \subset C$. As C is closed, by Proposition 8 one has that $x \in C$. Thus \mathcal{C} has a nonempty intersection, as our choice of C was completely free. \square

We turn to the product spaces now. Recall that for a general index set A indexing the components of the product X , it's best to view the elements of the product as functions $f : A \rightarrow \cup_{\alpha \in A} X_\alpha$ satisfying the property that $f(\alpha) \in X_\alpha$ for all $\alpha \in A$.

Lemma 16. *A net $\Phi : D \rightarrow X$ in a product space $X = \times_{\alpha \in A} X_\alpha$ converges to a point $x : A \rightarrow \cup X_\alpha$ iff $\pi_\alpha \circ \Phi$ converges to $x(\alpha)$ for any $\alpha \in A$.*

Proof. Φ converges to x means that Φ is eventually in every neighbourhood $U \ni x$. In particular it is eventually in every basic (in product topology) neighbourhood U , that satisfies $\pi_\alpha(U) = U_\alpha$ for some α and arbitrary open neighbourhood $U_\alpha \subset X_\alpha$ of $x(\alpha)$. Thus $\pi_\alpha \circ \Phi$ converges to $x(\alpha)$, as it is eventually in any neighbourhood of $x(\alpha)$.

The opposite implication gives us that for any $\alpha \in A$ the net $\pi_\alpha \circ \Phi$ is eventually in any x_α -neighbourhood $U_\alpha \subset X_\alpha$ of its limit x_α , and we will show that then Φ converges to $x : A \rightarrow \cup X_\alpha$, with $x(\alpha) = x_\alpha$. Here we really need to use that any open set $U \subset X$ satisfies $\pi_\alpha(U) = X_\alpha$ for all but finitely many α 's. Say, $\pi_{\alpha_i}(U) = U_{\alpha_i} \neq X_{\alpha_i}$ for $1 \leq i \leq n$. As $\pi_{\alpha_i} \circ \Phi$ is eventually in any U_{α_i} , there exists $\omega_i \in D$ such that $\pi_{\alpha_i}(\Phi(\tau)) \in U_{\alpha_i}$ as soon as $\tau \geq \omega_i$. As D is directed, there exists $\omega_U \geq \omega_i$ for $1 \leq i \leq n$. Thus $\Phi(\tau) \in U$ as soon as $\tau \geq \omega_U$, i.e. Φ is eventually in U . \square

Finally, we can prove Tychonoff's theorem.

Theorem 17. *(Tychonoff) The product $X = \times_{\alpha \in A} X_\alpha$ of an arbitrary collection of compact spaces X_α is compact.*

Proof. Let $f : D \rightarrow X$ be a universal net in X . The composition $\pi_\alpha \circ f$ is a universal net in X_α , by Lemma 13. Hence it converges to, say, x_α , by Proposition 15. Therefore f converges to the point $x : A \rightarrow \cup X_\alpha$ s.t. $x(\alpha) = x_\alpha$, by Lemma 16. Hence X is compact by Proposition 15. \square

References

- [1] G. E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.