NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2007–2008

MAS 436 – Topology

November 2007 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains THREE (3) questions and comprises THREE (3) printed pages.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a FRESH page of the answer book.
4. This IS NOT an OPEN BOOK exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

Question 1. (25 marks)
Let $p_1(X,Y),\dots,p_n(X,Y)$ be polynomials in X, Y. Let $S \subset \mathbb{R}^2$ be defined as

$$S = \{(x,y) \in \mathbb{R}^2 \mid 0 = p_1(x,y) = \cdots = p_n(x,y)\}.$$

Consider S as a subspace of $\mathbb{R}^2$ endowed with the usual metric topology.

(i) Is S always (i.e. for any n-tuple of polynomials p_i) connected?

(ii) Is S always closed?

(iii) Let π denote the projection on the coordinate Y. Is $\pi(S) \subseteq \mathbb{R}$ always closed? (Here $\mathbb{R}$ has the usual metric topology.)

Question 2. (40 marks)
Let X be a compact Hausdorff space, and $C(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$.

For any $x \in X$, let us define $\mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}$. Let us define the set of all such $\mathfrak{m}_x$

$$\tilde{X} = \{\mathfrak{m}_x \mid x \in X\} \subset 2^{C(X)}$$

and the corresponding mapping

$$\begin{aligned} \mu : X &\to \tilde{X} \\ x &\mapsto \mathfrak{m}_x. \end{aligned}$$

(i) Show that μ is injective.
Hint: use Urysohn's Lemma.

(ii) Show that the sets $U_f = \{x \in X \mid f(x) \neq 0\}$, for $f \in C(X)$, form a base of the topology of X.

(iii) Show that the sets $\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} \mid f \notin \mathfrak{m}\}$, for $f \in C(X)$, form a base of a topology on $\tilde{X}$.

(iv) Show that $\mu : X \to \tilde{X}$ is a homeomorphism.

Remark. For those who know enough algebra: $C(X)$ is a ring, the set endowed with operations $+$ and $\cdot$, as follows: $(f+g)(x) = f(x)+g(x)$, $(f\cdot g)(x) = f(x)g(x)$ for any $x \in X$ and any $f, g \in C(X)$. Then $\tilde{X}$ is the set of all maximal ideals in $C(X)$ and, as a topological space, is known as the *maximal spectrum* of $C(X)$.

Question 3. (35 marks)

A space X is called *locally compact* if each $x \in X$ has a neighbourhood contained in a compact subset.

(i) Show that the following three conditions are equivalent.

  (a) X is locally compact;

  (b) each $x \in X$ has a neighbourhood U such that its closure $\overline{U}$ is compact;

  (c) X has a basis $\mathcal{B}$ of open sets U such that $\overline{U}$ is compact for any $U \in \mathcal{B}$.

(ii) Let X, Y be locally compact Hausdorff spaces. Consider a continuous map $f : X \to Y$ such that $f^{-1}(U) \subseteq X$ is compact for any compact $U \subseteq Y$. (Such f is called *proper*). Show that f is a closed map.

END OF PAPER