Motor selection via impedance-matching for driving nonlinearly damped, resonant loads

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ARTICLE INFO

Article history:
Received 8 December 2009
Accepted 24 May 2010
Available online xxxx

Keywords:
Motor selection
Optimal transmission
Flapping wings or fins
Biomimetic robotics

ABSTRACT

This paper presents a novel method for selecting the most appropriate motor–gearhead combination for nonlinearly damped, resonant loads. The method extends to the nonlinear damping case the impedance-matching condition which is used to guarantee a maximum power transfer in linear network theory. In particular, the method is applicable in general to resonant loads where the damping is an odd and memoryless nonlinear function of the velocity. This condition is very common in biomimetic robotics, in particular when designing propulsion systems based on flapping appendages, such as wings or fins, in viscous fluids, such as air or water. The method is graphical in nature and is based on a power vs. impedance-mismatch factor. Such a factor is function of the ratio of the motor armature resistance to the load equivalent resistive impedance reflected at the motor armature, where the latter is nonlinear and depends on the desired kinematics as well as on the transmission ratio and efficiency. The method allows comparing, for a given desired appendage kinematics, all motor–gearhead combinations at once while taking into account all possible constraints such as maximum current, power, and torque.

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1. Introduction

Electric motors are pervasive and can be found in nearly all modern machines. As the appropriate choice of the electric motor and the relative mechanical transmission directly affects the performance of the whole system, various methods have been presented to guide the Engineer during the motor-transmission selection [26,27,9,21,8]. Most industrial tasks require a motor to routinely perform work against typically inertial and resistive loads. Although this is definitely the most common loading condition for motors, other forms of loading are also possible which call for different selection methods. One example is the nonlinear, viscous damping generated by flapping fins and wings against water and air.

Recent years have witnessed an increase of research efforts in what is generally referred to as biomimetic robotics. Attracted by the unmatched performance of living systems, roboticists have started applying design principles drawing inspiration from biological evidence. In particular, agility and maneuverability in water or in air of living swimmers and fliers has inspired the development of an increasing number of so-called Micro Underwater Vehicles (MUVs) and Micro Aerial Vehicles (MAVs) [1,14,13]. Besides bio-inspired sensing capabilities and neuro-inspired forms of controllers [15], there has been a technological push towards the development of biomimetic forms of propulsion, with particular emphasis on flapping fins and wings. Both in air and in water, flapping locomotion is superior to other forms of propulsion especially at lower speeds. Unparalleled by man-made vehicles, animals such birds, bats, insects and also negatively buoyant fishes are in fact capable of fast forward motion as well as hovering, which is one of the most energetically challenging forms of locomotion since it cannot exploit the accumulated kinetic energy of the body as in forward swimming/flying [28,3].

Efficient power usage is fundamental for the development of flapping propellers. One of the limits to flapping propulsion, also faced by living systems especially at larger size, is represented by the inertia of the wings/fins. The need to periodically accelerate/decelerate the inertia of the appendices poses serious constraints to the flapping modality. Although the primary interest is doing work against the fluid, as this directly translates into production of lift and thrust forces, it is not uncommon that accelerating/decelerating wings/fins at relatively high frequencies might require much larger inertial torques than damping ones. This would lead to oversize muscles (and actuators for artificial systems).

A first biological observation concerns the fundamental role played by resonance, physiologically induced by the structural elasticity typically inherent to the biomechanical properties of muscles and, for insects, of the external cuticle. The presence of elastic components for storage and release of the kinetic energy of the wings/fins avoids the unnecessary waste of inertial power [28]. Indeed, such a principle has been successfully adopted in the design of flapping wings and fins for robotic applications [30,6,2].
A second biological observation, which is key to the proposed selection method, is that the wings/fins kinematics observed in nature can, with a very good approximation, be considered harmonic. In particular, throughout this paper, this will be referred to as the sinusoidal regime assumption.

Based on the above considerations, this paper presents a novel method for selecting the appropriate motor–gearhead combination in order to directly drive a resonant mechanical system characterized by a typically nonlinear aero/fluid-dynamic damping. The nature of the nonlinearity is better detailed in the next section and, although the calculations are based on this particular form of nonlinear damping, the method is applicable whenever the nonlinearity is memoryless and odd. The proposed method is based on the matching impedance principle, a well known condition for linear networks which allows the maximum transfer of power from a fixed-voltage source to a load. The method is extended to the case of nonlinear resistive loads exploiting the sinusoidal regime condition and analytical techniques which are usually referred to ‘linear approximant’ or ‘describing function’ methods [11,22,23].

Finally, numerical tests are also presented in support of the sinusoidal regime assumption.

2. Fluid dynamical nonlinear damping

From a fluid dynamical perspective, the dynamics of hovering are qualitatively similar in air as much as underwater and insects, birds, bats and negatively buoyant fishes can be treat as a single functional group [3].

When a fin or a wing is moved relative to the surrounding fluid, energy is transferred to the fluid and reaction forces arise. In principle, the force distribution on the fin/wing as well as flow field of the surrounding fluid may be derived from the Navier–Stokes equations. In practice, numerical approaches (computational fluid dynamics) are the only viable ones for accurate solutions. Nevertheless, classical, reliable simplifications are based on the assumption of steady or quasi-steady flow [5,18]. Such models are based on quasi-steady blade element analysis where the wing/fin is assumed to be divided into a finite number of strips and each is analyzed independently.

For a wing/fin of length $L$, at distance $r$ from the fulcrum, consider sections of infinitesimal area $c(r) dr$, where $c(r)$ is the wing chord which determines the geometric profile. For each section, the instantaneous drag force is

$$dD = \frac{1}{2} \rho C_D (r \omega_0)^2 \text{sign}(\omega) c(r) dr$$

where $\rho$ is the density of the fluid (for air $\rho_{\text{air}} = 1.2 \text{ kg/m}^3$ and for water $\rho_{\text{water}} = 10^3 \text{ kg/m}^3$), $C_D$ is the adimensional drag coefficient, and $r \omega_0$ is the linear velocity of the section. $C_D$ depends on the angle of attack (i.e. inclination of the wing/fin with respect to the fluid velocity), therefore is in principle time dependent and can be averaged out throughout the motion [29]. $C_D$ is close to zero for small angles of attack and monotonically increases up to a maximum value as the angle of attack approaches $\pi/2$. Maximal values range from 3.5 for some insects [10], to 0.4 for hummingbirds [28], to 2 for some fishes [3].

On any section, a drag torque $dB = r dD$ is induced by the drag force, i.e.

$$dB = \frac{1}{2} \rho C_D (r \omega_0)^2 \text{sign}(\omega) c(r) dr$$

For a given wing profile $c(r)$, the above equations can be integrated over the whole wing length, leading to:

$$B(\omega) = \int _0 ^{\pi} dB = B_0 \omega_0^2 \text{sign}(\omega)$$  \hspace{1cm} (1)

with the torque damping coefficient $B_0$ defined as:

$$B_0 = \frac{1}{2} \rho C_D \int _0 ^{\pi} r^2 c(r) dr$$

where the integral depends on the particular wing/fin profile and its analytical evaluation is given in [29] for various wing shapes. For simulation purposes, numerical values for parameters were derived from [12] in the case of a box-fish and are summarized in Table 1.

The nonlinear damping $B(\omega)$ in (1) is therefore memoryless, since does not depend explicitly on time, and odd, meaning that $B(-\omega) = -B(\omega)$.

3. Sinusoidal regime assumption

Biological observations [29,17,25] provide evidence that: (i) mechanical resonance is the mechanism through which animals efficiently flap appendages; (ii) the resulting kinematics is with good approximation sinusoidal. Inherent elasticity in the muscular-skeletal structure of animals plays a key role in storing elastic strain energy which, especially in cyclic movements such as flapping appendages, can be released to efficiently enhance the force, work or power output.

Based on such observations, the sinusoidal regime assumption will allow estimating power requirements at steady-state: for a given stroke-angle $\alpha_0$ and a given flapping frequency $f_0$, the angular wing motion $\theta(t)$, angular wing velocity $\dot{\theta}(t)$ and the angular wing acceleration $\ddot{\theta}(t)$ in sinusoidal regime (denoted by the ~ symbol) can be expressed as

$$\ddot{\theta} = \theta_0 \sin(2\pi f_0 t)$$ \hspace{1cm} (2)

$$\dot{\theta} = \Omega_0 \cos(2\pi f_0 t)$$ \hspace{1cm} (3)

$$\ddot{\theta} = -2\pi f_0 \theta_0 \sin(2\pi f_0 t)$$ \hspace{1cm} (4)

where the for the angular speed amplitude $\Omega_0$, the following definition will be conveniently adopted:

$$\Omega_0 \equiv 2\pi f_0 \theta_0$$ \hspace{1cm} (5)

At sinusoidal regime, the nonlinear damping torque $B(\dot{\theta})$ defined in (1), can be expressed in terms of an equivalent proportional damping $B_1 \dot{\theta}$, where $B_1$ is defined to provide the same average power (over a period $T = 1/f_0$):

$$\frac{1}{T} \int _0 ^{T} B_1 \dot{\theta}^2 dt = \frac{1}{T} \int _0 ^{T} B(\dot{\theta}) \dot{\theta} dt$$

i.e.

$$\int _0 ^{T} B_1 \dot{\theta}^2 dt = \int _0 ^{T} B_0 \dot{\theta}^2 \sin(2\pi f_0 t) dt$$

With change of variable $(2\pi f_0 t \rightarrow x)$, this can conveniently computed over a quarter of period

<table>
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<th>Description</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<tr>
<td>damping</td>
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<td>Nm$^2$ s$^{-1}$</td>
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<tr>
<td>Frequency</td>
<td>$f_0$</td>
<td>4</td>
<td>Hz</td>
</tr>
</tbody>
</table>

Table 1 Parameters used for simulation. Numerical values were derived from [12].

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[1] For underwater swimming animals and vehicles, drag refers to the whole body effect while thrust is considered for the fin only, therefore they are related by the body-area to foil-area ratio [16]. In this paper, only the fluid dynamical effects on the wing or fin are considered and drag shall be equivalent to thrust.

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Please cite this article in press as: Campolo D. Motor selection via impedance-matching for driving nonlinearly damped, resonant loads. Mechatronics (2010), doi:10.1016/j.mechatronics.2010.05.008
The rotor and its acceleration of the rotor, $\omega$, is expressed as:

$$B_1 \int_{-\pi/2}^{\pi/2} \Omega_0^2 \cos^2(x) \, dx = B_0 \int_{-\pi/2}^{\pi/2} \Omega_0^3 \cos^3(x) \, dx$$

and analytically evaluated as:

$$B_1 = B_0 \frac{8}{3\pi} \Omega_0^2$$

Therefore, at sinusoidal regime, the damping torque approximated by

$$\bar{B}(\dot{\omega}) = B_0 \frac{8}{3\pi} \Omega_0 \dot{\omega}$$

with a peak torque

$$|\bar{B}(\dot{\omega})| = B_0 \frac{8}{3\pi} \Omega_0^3$$

where $|\cdot|$ denotes the amplitude of a sinusoidal variable. Consequently, the power dissipated against drag is

$$P_{\text{drag}} = \bar{B}(\dot{\omega})\dot{\omega} = B_0 \frac{8}{3\pi} \Omega_0^3 \dot{\omega}^2$$

with a peak power (equivalent to the power amplitude)

$$P_{\text{avg}} = |\bar{P}_{\text{drag}}| = B_0 \frac{8}{3\pi} \Omega_0^3$$

4. DC motor selection

4.1. Permanent DC motor model

The electro-mechanical model of a gearless DCM [7], considering a steady-state\(^2\) driving voltage $V$ and current $I$, can be summarized in the following equations:

$$\begin{cases} V = R_0 I + K_a \omega_m \\ K_a I - \tau_m = J_m \omega_m + b_i \omega_m \end{cases}$$

(11)

where $\omega_m$ and $\omega_m$ are, respectively, the angular speed and the angular acceleration of the rotor, $R_0$ is the electrical resistance of the armature, $K_a$ is armature constant, $J_m$ is the moment of inertia of the rotor and $b_i$ is the damping constant due to the internal friction (motor bearings), $\tau_m$ represents a generic external torque exerted directly on the shaft of the rotor.

4.2. Gearhead characteristics

The gearhead, as other forms of transmission, is an element used to optimally match the motor characteristics with a given load. The gearhead is often treated as an ideal gear ratio $N_g$. Although this might be acceptable in many cases, especially for high gear ratio it might lead to underestimates of the required power.

In this paper, together with the gear ratio $N_g$ also the gear efficiency $\eta_g$ will be considered, leading to the following relations between torque $\tau_m$ and speed $\omega_m$ at the motor and torque $\tau$ and speed $\omega$ at the load:

$$\begin{align} \omega_m &= N_g \omega \\
\tau &= N_g \eta_g \tau_m \end{align}$$

(12)\hspace{1cm}(13)

in the gearless condition, i.e. when the motor directly drives the load, then such parameters can be conveniently set to $N_g = 1$ and $\eta_g = 1$.

4.3. Overall system equations

A generic resonant load with inertia $J_p$, stiffness $K_p$ and nonlinear damping as in (1) is described by

$$J_p \ddot{\omega} + B_0 \omega^2 \text{sign}(\omega) + K_p \omega = \tau$$

(14)

where $\tau$ as in (13), is the driving torque at gearhead output. By combining Eqs. (11)-(14), the overall system can be described as

$$\begin{cases} V = R_0 I + K_a N_g \omega \\ \eta_g N_g K_a I = (\eta_g N_g^2 J_m + J_1) \dot{\omega} + (\eta_g N_g^2 b_i + B_0 \omega \text{sign}(\omega)) \omega + K_p \theta \end{cases}$$

(15)

4.4. Overall system equations at sinusoidal regime

If the stiffness $K_p$ is selected to ensure that the whole systems resonates at frequency $f_0$, i.e.

$$K_p = (2\pi f_0)^2 \left(\eta_g N_g^2 J_m + J_1\right)$$

(16)

then at sinusoidal regime, using (2)-(4), the inertial torque and the elastic torque will balance one another

$$\left(\eta_g N_g^2 J_m + J_1\right) \dot{x} + K_p \theta = 0$$

(17)

reducing the final set of equations to:

$$\begin{cases} \ddot{V} = R_0 I + K_a N_g \dot{\omega} \\ \eta_g N_g K_a I = (b_0 + \frac{8}{3\pi} B_0 \Omega_0^2) \dot{\omega} \end{cases}$$

(18)

where $b_0 = \eta_g N_g^2 b_i$. Considering a sinusoidal input voltage $\ddot{V} = V_0 \cos(2\pi f_0 t)$, armature current $I = I_0 \cos(2\pi f_0 t)$ and angular velocity as in (3), after simplifying the cos($2\pi f_0 t$) dependence, voltage amplitude $V_0$ and current amplitude $I_0$ can be expressed as a function of $\Omega_0$:

$$\begin{cases} V_0 = \frac{R_0}{\eta_g N_g} \left(b_0 + \frac{8}{3\pi} B_0 \Omega_0^2\right) \Omega_0 + K_a N_g \Omega_0 \\ I_0 = \frac{R_0}{\eta_g N_g} \left(b_0 + \frac{8}{3\pi} B_0 \Omega_0^2\right) \Omega_0 \end{cases}$$

(19)

which guarantees a one-to-one relationship for all $V_0, I_0, \Omega_0 \geq 0$ (as amplitudes are non-negative by definition).

4.5. Power estimates at sinusoidal regime

An instantaneous power balance can be obtained from (18) as

$$\bar{P} = R_0 I^2 + \frac{1}{\eta_g} \left(1 + \frac{b_0}{\pi B_0 \Omega_0}\right) \frac{8}{3\pi} B_0 \Omega_0^2 \omega^2$$

(20)

which can be rewritten to highlight the power dissipated against mechanical damping (input electrical power minus electrical losses):

$$\bar{P}_{\text{mech}} \overset{\text{def}}{=} \bar{P} - R_0 I^2 = \frac{1}{\eta_g} \frac{1}{\eta_g} \bar{P}_{\text{drag}}$$

(20)

where $\bar{P}_{\text{drag}}$ is as in (9) and $\eta_g$ is defined as

$$\eta_g = \left(1 + \frac{b_0}{\pi B_0 \Omega_0}\right)^{-1}$$

(20)

to take into account the mechanical power dissipated against shaft friction instead of fluid dynamical drag. Therefore, the mechanical power $\bar{P}_{\text{mech}}$ is dissipated partly in the gearhead (accounted for by $\eta_g$) and partly in the shaft friction (accounted for by $\eta_g$) before being dissipated against drag.
While efficiencies $\eta_g$ and $\eta_h$ can be very close to 1, there is an intrinsic limit to the mechanical power that can be delivered in relation to the electrical losses. The maximum power transfer theorem for linear networks states that, for a given nominal input voltage, the maximum mechanical power delivered to the load cannot exceed the electrical losses, leading to a maximum 50% efficiency which can only be achieved in the case of impedance-matching condition. A similar result can be derived here. Substituting the second equation into the first equation of (18) leads to:

$$\bar{V} = R_0 \bar{I} + R_{\text{mech}} \bar{I}$$

where

$$R_{\text{mech}} = \frac{\eta_g \eta_h N_0^2}{L_0} \frac{K^2}{2B_0} \Omega_0$$

is the *equivalent electrical impedance* of the mechanical damping, comprising both the drag and the friction at the shaft and gears, as seen at the armature.

The instantaneous mechanical power is therefore

$$P_{\text{mech}} = \bar{V} \bar{I} = R_0 \bar{I}^2 - \frac{R_0 R_{\text{mech}}}{(R_0 + R_{\text{mech}})^2} \bar{V}^2 = \mu \left( 1 - \frac{1}{2} \frac{\bar{V}^2}{R_0} \right)$$

where

$$\mu = 4 \frac{R_0 R_{\text{mech}}}{(R_0 + R_{\text{mech}})^2} \geq \frac{R_{\text{mech}}}{R_0}$$

is the impedance-mismatch factor. Note that $0 < \mu \leq 1 \forall R_0, R_{\text{mech}} > 0$ and also that $\mu = 1$ if and only if $R_0 = R_{\text{mech}}$, meaning that the power dissipated across $R_{\text{mech}}$ equals the power dissipated across $R_0$, i.e. a 50% efficiency. In the best case scenario (i.e. impedance-matching condition: $R_{\text{mech}} = R_0$) the total input power is $\bar{V}^2/(2R_0)$ and only half of it can be transferred to the mechanical load.

Eq. (20) can now be conveniently rearranged as

$$\frac{\eta_g \eta_h}{2} \frac{1}{R_0} \frac{\bar{V}^2}{\mu} = \frac{P_{\text{drag}}}{\bar{V}}$$

or, considering that $\bar{V} = V_0 \cos(2\pi f_0 t)$ and $P_{\text{drag}} = P_{\text{drag}} \cos(2\pi f_0 t)$, in terms of amplitudes

$$\eta_g \frac{V_0^2}{2R_0} = \frac{P_{\text{drag}}}{\bar{V}}$$

where $P_{\text{drag}}$ is given in (10).

It should first be noted that (23) is equivalent to the first of (19) as it defines a unique input voltage amplitude $V_0$ (non-negative by definition) for any desired angular velocity amplitude $\Omega_0$ (also non-negative by definition). The advantage is that (23) provides an interpretation in terms of power and leads to a graphical representation useful for motor selection, as we shall see next.

For a given desired kinematics, i.e. a desired stroke–angle $\theta_0$ at fixed frequency $f_0$ which uniquely defines $\Omega_0$ from (5), the power $P_{\text{drag}}$ to be dissipated against drag is the minimum amount that the motor should be able to handle. In fact, the power required from the motor can be even larger in case of impedance-mismatch $\mu < 1$, as indicated by the right-hand side of (23). The left-hand side of (23) represents the maximum available power for a given fixed input voltage amplitude $V_0$. Recalling that the maximum transferable power to the load is one half of $V_0^2/(2R_0)$, the available power is further reduced by the gear inefficiencies $\eta_g$ and by the shaft friction $\eta_h$. It is therefore advantageous to select motors with optimal impedance match ($R_{\text{mech}}$ as close as possible to $R_0$) so to minimize the required power and not to oversize the motor itself.

Impedance-ratio is also the key to an effective graphical representation. The required power, i.e. the right-hand side of (23), can be plotted in a power vs. impedance-ratio plot independently of any motor–gear selection, and is represented by a family of curves (which we shall refer to as ‘drag curves’) parameterized by $P_{\text{drag}}$ which in turn only depends on $\Omega_0$. The shape of such curves is solely dictated by $\mu^{-1}$, function of the impedance-ratio $R_{\text{mech}}/R_0$ as defined in (22), and all assume minimum value $P_{\text{drag}}$ when $R_{\text{mech}}/R_0 = 1$. For example, in the sketch in Fig. 1 two drag curves (dashed line) are drawn corresponding to drags $P_{\text{drag}}$ and $P_{\text{drag}}$ which are relative, respectively, to two different angular speeds $\Omega_e$ and $\Omega_e$.

As for the left-hand side of (23), let us consider for just a moment the case of linear damping, meaning that the once a motor–gear combination is selected then the $R_{\text{mech}}/R_0$ ratio remains constant. For this linear case, in a power vs. impedance-ratio plot, the left-hand side of (23) would simply be a vertical line (hereafter referred to as ‘motor line’) parameterized with the input voltage amplitude $V_0$. In Fig. 1 the vertical solid line represents the locus of points with a fixed impedance-ratio (greater than one, in this specific example) and power proportional to $V_0$. The two drag curves relative to $\Omega_e$ and $\Omega_e$ intersect the vertical line in two points which are relative to input voltage amplitudes $V_e$ and $V_e$. This graphical correspondence defines a one-to-one relationship between $V_0$ and $\Omega_e$.

Of course, we are interested in nonlinear damping such as (1) in which case the impedance-ratio does depend on the operating speed $\Omega_0$. The left-hand term in (23) can still be represented in a power vs. impedance-ratio plot, the only difference is that it will no longer correspond to a vertical line. In fact, as we shall see in the numerical study in next section, when represented in logarithmic scale, each motor–gear selection can be practically considered as a very steep line, especially when compared with the drag curves. Different motor–gear selections can be represented in the same power vs. impedance-ratio plot providing an immediate means of comparison for selecting the best option.

5. Numerical study

In this section, an optimal motor–gearhead combination shall be selected for driving at resonance the wing model presented in Section 2. Among various possibilities of commercial motors and gearheads, the miniature devices from Faulhaber [19] were chosen because of the availability of electro-mechanical parameters for simulations. In particular the motors listed in Table 2 were considered in combination with the gearheads listed in Table 3. As for a matter of nomenclature, the motors in Table 2 will be indicated with numbers 1, ..., 5 (relative to their ID) while letters ‘b’, ’c’, ’g’ shall be used to refer to a particular gear ratio, the letter ‘a’ is reserved to indicate a gearless combination, in which case an ideal gear ratio $N_{\text{gear}} = 1$ and efficiency $\eta = 100\%$ will be assumed. Note

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![Fig. 1. Power vs. impedance-ratio plot in the linear case, i.e. when the impedance-ratio $R_{\text{mech}}/R_0$ remains constant.](image-url)
Table 2

Electro-mechanical parameters relative to five motors from [19]. $V_n$ is the nominal voltage, $R_o$ is the armature resistance, PWR is the maximum rated power, $n_0$ is the no-load speed, $I_0$ is the no-load current, $T_{max}$ is the torque at stall, $T_{friction}$ is the friction relative to the no-load speed $n_0$, $K_a$ is the armature constant, $J_m$ is the rotor inertia while $n_{max}$, $T_{max}$ and $I_{max}$ are the maximum ratings for, respectively, speed, torque and current.

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<tr>
<th>ID</th>
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<th>3</th>
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<td>0.12</td>
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<td>12,000</td>
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</tbody>
</table>

Table 3

Mechanical parameters relative to three sets of gearheads to be used in combination with motors from [19].

<table>
<thead>
<tr>
<th>Gearheads</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
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</thead>
<tbody>
<tr>
<td>06/1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
<td>4096</td>
</tr>
<tr>
<td>$n_0$</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>55</td>
<td>48</td>
</tr>
<tr>
<td>$T_{friction}$</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>mNm</td>
</tr>
<tr>
<td>Weight</td>
<td>2.0</td>
<td>2.8</td>
<td>3.4</td>
<td>4.0</td>
<td>4.4</td>
<td>5.0 g</td>
</tr>
<tr>
<td>08/1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
<td>4096</td>
</tr>
<tr>
<td>$n_0$</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>55</td>
<td>48</td>
</tr>
<tr>
<td>$T_{friction}$</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>mNm</td>
</tr>
<tr>
<td>Weight</td>
<td>2.9</td>
<td>3.8</td>
<td>4.6</td>
<td>5.4</td>
<td>6.3</td>
<td>7.1 g</td>
</tr>
<tr>
<td>10/1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
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<td>80</td>
<td>70</td>
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<td>48</td>
</tr>
<tr>
<td>$T_{friction}$</td>
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<td>15</td>
<td>54</td>
<td>100</td>
<td>100</td>
<td>mNm</td>
</tr>
<tr>
<td>Weight</td>
<td>6.0</td>
<td>7.0</td>
<td>8.0</td>
<td>10</td>
<td>11</td>
<td>13 g</td>
</tr>
</tbody>
</table>

Fig. 2 shows the power vs. impedance-ratio plot for all the motors in Table 2 combined with the relative gearheads in Table 3 and for the power drag requirements as in Section 2. The drag curves were drawn for three values of $\Omega_0$ corresponding to $40^\circ$ (bottom drag curve), $50^\circ$ (middle drag curve) and $60^\circ$ (top drag curve) stroke-angle amplitudes ($h_0$). The maximum angular speed amplitude $\Omega_0^\max$ is relative to a $60^\circ$ ($\pi/3$) stroke-angle and equals

$$\Omega_0^\max = 2\pi h_0 \frac{\pi}{3} \approx 26.3 \text{ rad/s}$$

For each motor–gearhead combination, 60 equally spaced (each corresponding to 1° increment in stroke-angle) possible operating angular speed amplitudes $\Omega_0 \in [0, \Omega_0^\max]$ were considered and for each operating speed the required input voltage $V_n$ and current $I_0$ amplitudes were computed via (19).

Although solutions of (19), large values of input voltage $V_n$ or current $I_0$, input power $P_{drag}$ or speed $\Omega_0$ will likely exceed some intrinsic limit of the motor–gearhead combination. For this reason, for each motor–gearhead combination, only those operating conditions would be represented in the power vs. impedance-ratio plot which satisfy at the same time all the constraints as given in Table 4. Fig. 2 represents for each motor–gearhead combination such admissible operating conditions. As mentioned in previous section, in a logarithmic scale the left-hand side of (23) appears as a steep (almost vertical) curve for each motor–gearhead combination.

Fig. 2 allows to compare at once the performance of all the available motor–gearhead combinations. It can be appreciated that not all the motor lines reach the $60^\circ$ drag curve. For example, motors 1 and 2 (which are the smallest in weight, see Table 2) with all the possible gearhead combinations (a–g) have not enough power to reach the $60^\circ$ drag curve. The combination 2c (motor 2 with a 16 gear ratio gearhead) shows the most favorable impedance-ratio

![Image of Table 2](image)

![Image of Table 3](image)

![Image of Fig. 2](image)

Please cite this article in press as: Campolo D. Motor selection via impedance-matching for driving nonlinearly damped, resonant loads. Mechatronics (2010), doi:10.1016/j.mechatronics.2010.05.008
(i.e. the closest to 1) and is the only combination involving motor 2 that can provide more than 50° stroke-angle.

In fact the only combinations that can reach the 60° drag curve are the 3c, 4c, 5b and 5c. Due to large impedance-mismatch, combination 5b and 5c can indeed produce the desired output \( \Omega_{\text{aux}} \) but at a cost of a larger required power (\( \approx 100 \text{ mW} \), note that the motor 5 is also the most powerful of the series). On the other hand, combinations 3c and 4c are equally able to produce the desired kinematic output \( \Omega_{\text{aux}} \) with lower effort (\( \approx 70 \text{ mW} \)) as the impedance-mismatch factor is close to 1. Of the two, combination 3c is to be preferred because of a smaller weight (6.5 g for motor 3 vs. 11 g for motor 4, for both combinations the gearhead c would add 7 g).

**Remark.** It is worth noting that, since the slopes of the motor lines are negative, motor lines that start with an impedance-ratio larger than 1 for small output speed, gain a better impedance-ratio (i.e. closer to 1) as speed increases. The impedance-ratio counts more at higher speed which is when most of the power is required.

6. **Accuracy of the sinusoidal regime hypothesis**

The Eqs. (18)–(23) are based on the sinusoidal regime hypothesis. Here numerical simulations are presented to assess the level of accuracy of such an assumption. In particular, the nonlinear ordinary differential equations (ODE) in (15) is numerically solved without the sinusoidal regime simplifying hypothesis. All numerical parameters required for the simulations can be found in Section 2 as well as in Tables 2 and 3, the stiffness \( K_t \) which ensures resonance at \( f_1 = 4 \text{ Hz} \) is determined as in (16).

To determine how indicative the proposed impedance-matching based method can be, motors 3, 4 and 5 (which proved powerful enough according to the proposed method to produce the desired output kinematics) were numerically tested with a variety of gear ratios. In particular, we simulated the motor performance with any integer gear ratio between 1 and 100, considering an ideal 100% efficiency for all ratios except for the actually available 4, 16, and 64 ratios, for which the efficiencies as in Table 3 were used instead.

Similarly to the framework of the maximum power transfer theorem, which holds for linear networks and which was here proved under the sinusoidal regime assumption, we also assumed a constant amplitude voltage \( V = V_p \cos(2\pi f_1 t) \) as an input to (15). The system was implemented in MATLAB (MathWorks Inc.) and the ODE was solved with the built-in ode45() function. Since we were only interested in the steady-state solutions, we started with arbitrary initial conditions \( \theta(0) = 0 \) and \( \omega(0) = 0 \) and simulated the system for exactly one period of time \( T = 1/f_1 \). Then the simulation was repeated using the final values \( \theta(T) \) and \( \omega(T) \) as initial conditions for the new simulation. This step was repeated until the relative error between the current solutions and the solutions provided by the previous simulation step dropped below 0.1% (maximum error throughout the period).

Fig. 3 shows the response for motors 3, 4, and 5. As the motors are different, also different fixed input voltage amplitudes were used (2 V, 5 V and 4 V, respectively, for motor 3, 4, and 5), heuristically chosen to provide an approximately 60° output stroke-angle for the 3c, 4c, 5b and 5c cases. The vertical lines in Fig. 3 indicate the actually available gear ratios 4, 16, 64 as well as the gearless case \( N_g = 1 \).

The situation depicted in Fig. 2 is confirmed by such numerical simulations. For each plot in Fig. 3, it can be recognized what the optimal gear ratio would be for each case (e.g. for motor 5, \( N_g = 10 \) would give the maximum \( \Omega_0 \) output). Although the actually available gear ratios never perfectly match the optimal ones, it can be recognized that 3c and 4c (corresponding to the third vertical line, respectively, in the bottom and middle plot of Fig. 3) are very close to optimality. Also note that since for the available gear ratios the actual non-ideal efficiency was used, their output is slightly smaller than expected from the whole trend of ideal gears.
Fig. 4 shows the phase-space $\omega$ vs. $\theta$ plot (top graph) of the actual steady-state solutions of (15) superimposed with pure sinusoidal solutions assumed under the sinusoidal regime hypothesis. As expected there is very little difference. In particular, relative errors between the actual and the pure sinusoidal solutions is within 5% for the $\omega$ and within 3% for $\theta$ (middle plot in Fig. 4). The output torques are also reported in the bottom plot of Fig. 4. In particular, the nonlinear torque as in (1) superimposed with the linear approximant as in (7) are presented as well as the elasto-kinetic (EK) contribution. The latter is in fact the torque which is not compensated for by the resonant system. The actual steady-state solutions of (15) are not in fact purely sinusoidal, although periodic, meaning that higher harmonics are also present. The mechanical resonance is capable of perfectly balancing the elastic and inertial (i.e. kinetic) torques only for the first harmonic (at frequency $f_0$). For higher harmonics (i.e. frequencies $nf_0$, with $n > 1$) the inertial torque will be larger than the elastic one, in particular

$$-(2\pi n f_0)^2 \left( \eta n^2 \frac{J_g}{J_i} + J_i \right) + K_i \neq 0, \quad \forall n > 1 \quad (24)$$

It should be noted that since $B(\omega)$ in (1) is a odd nonlinearity, i.e. $B(-\omega) = -B(\omega)$, the steady-state periodic solutions will only contain odd frequencies. There is no contribution for the second harmonic and the first higher harmonic is in fact the third. This can also be appreciated in the bottom plot of Fig. 4 where the elasto-kinetic (EK) torque is clearly periodic of period $1/(3f_0)$.

7. Conclusion

This paper presents a novel method for selecting the most appropriate motor–gearhead combination for driving nonlinearly damped loads at resonance. This loading condition is common in biomimetic robotics, when designing propulsion systems based on flapping appendages, such as fins or wings, in a viscous fluid such as water or air. Specifically to this condition, a typically non-linear, quadratic damping is considered. The power estimates derived in this paper allow representing the problem on a power vs. impedance-mismatch plot, as in Fig. 2, where all available motor–gearhead combinations can be compared at once with respect to the output power requirements.

To derive power estimates, we first determine the dynamic equations for the full system (15) taking into account: (i) the electro-mechanical equations of the DC motor; (ii) the gearhead characteristics (in particular the transmission efficiency); (iii) the load, which consists of the inertia and nonlinear damping of the wing/fins as well as of the elastic properties of the coupling mechanism.

Such a loading condition reflects an important biomimetic aspect: resonance. It is in fact biological evidence that the efficiency behind locomotion based on wings/fins-flapping is largely due to the elastic properties of muscles, or to the external cuticle in the case of insects, which induce structural resonance when coupled with the inertia of the flapping appendages themselves. The presence of such elastic components for storage and release of kinetic energy avoids in fact unnecessary waste of inertial power (required to accelerate/decelerate wings or fins).

The theory of linear networks suggests that, to maximize power transmission when driving a resonant load, the optimal gear ratio should be chosen so that the resistive impedance of the load, reflected at the actuator, matches the impedance of the actuator itself. The proposed method extends the linear case to the nonlinear one based on a second biological observation: the wings/fins kinematics observed in nature can be considered harmonic with a good approximation.

Considering steady-state solutions which are approximately sinusoidal allows us to remarkably simplify analytical calculations and derive approximate solutions which closely match the numerical solutions of the original system, e.g. see Fig. 4 (top). In particular, numerical steady-state solutions derived from the full nonlinear dynamics (15) provide support for the simplifying sinusoidal regime assumption, as the difference between the numerical steady-state solutions and pure sinusoids is lower than 5% and 3% for the angular speed and position, respectively, for the optimal motor–gearhead combination.

The fact that such a good matching also holds at large strokes (±60°) can be explained with two main observations: (i) we are not over-simplifying the system via some mere linearization but we are maintaining its central feature, i.e. the nonlinear damping; (ii) for the specific loading condition of interest (14) the nonlinearity is only in the damping and not in the elastic or inertial terms.

The latter observation has profound practical implications since, in order to fully exploit resonance, an elastic system should be designed to perfectly4 couple with the inertia to satisfy Eq. (16) and this is conveniently done when both inertial and elastic terms are linear. For the inertial term to maintain linearity at large strokes, direct driving of the load is advised. As also underlined by Baek, Ma and Fearing [2], commonly employed transmissions such as slider–crank mechanisms suffer from fundamental geometric nonlinearities, directly reflected in the inertial term. Such systems, when coupled with springs for energy recovery, fail at fully exploiting resonance due to the inherent kinematic nonlinearity.

Linearity requirements for the elastic term at large deformations call for a careful selection of both materials and structures. It is not uncommon, for large stroke requirements, that nonlinear elastic behaviors heavily affect the resonance and therefore the method proposed in this paper, e.g. previous work [24,4] on flapping wings reported and modeled resonant frequency shifts due to the so-called `softening' of piezoelectric actuators at higher fields. Although technologically challenging, achieving linearity of elastic structures over large range of deformations and frequencies is indeed feasible, as demonstrated by the various examples of resonant optical scanners5 currently used in laser applications [20]. To the author's knowledge, this is the only application where magnetic torque motors (with working principles similar to regular permanent magnet DC motors) are used to directly drive a resonant load, typically a mirror used to reflect a laser beam for optical scanning purposes. Mechanical resonance between the elastic structure (typically a torsional rod or a flexure joint [20]) and the inertia of the mirror is used to minimize the driving torque required from the motor. Nevertheless, an oscillating mirror (often resonating in vacuum) is different from a flapping wing for which nonlinear aerodynamic damping must be considered but, despite such important differences, resonant scanners demonstrate that it is indeed possible to efficiently6 and directly drive a resonant load at large angular displacements.

In conclusion, this paper provides a method for selecting DC motors for driving nonlinearly damped, resonant loads based on `quick' power estimates inspired by Weiss-Fogh's analysis [29], whose quotation is in order here:

4 It should be noted that such a 'perfect' coupling only holds for the first harmonic, i.e. for Eq. (17), where elastic and inertial term perfectly cancel each other at steady-state (a well known fact from second-order linear systems). This balance does not hold for higher harmonics, leading to the inequality (24). The inertial and elastic terms do have an effect for higher harmonics, resulting in a non-zero elasto-kinetic torque still required from the motor, as shown in Fig. 4 (bottom plot).

5 Also known as resonant galvanometers or ‘galvos’.

6 The efficiency resulting primarily in very compact active devices, see for example the SC-3 sub-miniature model from Electro-Optical Products Corporation (http://www.eopc.com/), where a 6 mm diameter, 1 mm thick aluminum mirror oscillates at 250 Hz with a ±5° angular stroke.
“Although entitled ‘Quick estimates’, this does not mean that the approach is superficial, but rather that a procedure has been devised whereby the flight performance of a given animal can be evaluated quantitatively as well as qualitatively on the basis of only a few accurate observational data and a minimum of computation.”

– verbatim from [29, p. 169]

Acknowledgement

This work is supported by the Academic Research Fund (AcRF) Tier1 (RG 40/09), Ministry of Education, Singapore.

References