Optimal Production and Rationing Policies of a Make-To-Stock Production System with Batch Demand and Backordering

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Abstract: In this paper, we consider stock rationing problem of a single-item make-to-stock production/inventory system with multiple demand classes. Demand arrives as a Poisson process with a randomly distributed batch size. It is assumed that the batch demand can be partially satisfied. The facility can produce a batch up to a certain capacity at the same time. Production time follows an exponential distribution. We show that the optimal policy is characterized by multiple rationing levels.

\textbf{Keywords:} Production; Inventory; Rationing; Markov Decision Process; Batch Demand

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Appendix: Proofs for Proposition 2

The following parts (from Lemma 2 to Lemma 5) are used to prove Proposition 2. Obviously, Lemma 2 is an immediate result from (9) and (10).

**Lemma 2.** For any nonnegative integer \( k \) and state \( x \), consider \( v(x + ie_i + ie_j) \), \( i = 0, \ldots, k \), \( 2 \leq l \leq N \). Define \( j^* = \min \{ \arg \min_{0 \leq i \leq l} v(x + ie_i + ie_j) \} \) and \( j^{**} = \min \{ \arg \min_{1 \leq i \leq j + k + 1} v(x + ie_i + je_i) \} \), then we have (1) \( v(x + ie_i + ie_j) \downarrow i \) for \( 0 \leq i \leq j^* \) and \( v(x + ie_i + ie_j) \uparrow i \) for \( j^* \leq i \leq k \), where \( \uparrow \) and \( \downarrow \) denote nondecreasing and nonincreasing, respectively. (2) \( j^{**} = j^* \) or \( j^{**} = j^* + 1 \).

To obtain the optimal structure of the production and rationing policy, we need only to prove that \( T \) preserves these structural properties (3) – (8). By (2), it suffices to show that if \( v \in V \), then \( c \in V \), \( T_{0v} \in V \), \( T_{ia}v \in V \) and \( T_{ia}v \in V \). Since \( c \) is similar as those defined in de Vericourt et al. [4], it has been proved therein that \( c \in V \). The proof for \( T_{ia} \) can follow that in Huang & Iravani [12] with slight revision. In the following part, we focus on proving \( T_{ia}v \in V \) \((2 \leq l \leq N)\) and \( T_{0v} \in V \), i.e., \( T_{ia}v \) and \( T_{0v} \) preserve properties (3) – (8). We first need the following lemmas (Lemma 3 to Lemma 5).

**Lemma 3.** Suppose \( n = \alpha \leq x \). For any demand class \( l \), \( 2 \leq l \leq N \), define

\[
i^*_1 = \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_i + (\alpha - i + 1)e_i) \},
\]

\[
i^*_2 = \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_i + (\alpha - i)e_i) \},
\]

\[
i^*_3 = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_i + (\alpha - i + 1)e_i) \},
\]

\[
i^*_4 = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_i + (\alpha - i)e_i) \},
\]

we have,

(1) If \( i^*_2 = i^*_3 = n \), then \( i^*_1 = i^*_4 = n \).
If $i^*_2 = i^*_3 = 0$, then $i^*_1 = i^*_4 = 0$.

(2) If $i^*_2 \neq n$ and $i^*_3 \neq 0$, then $i^*_1 = i^*_4 = i^*_1 + 1 = i^*_2 + 1$.

Lemma 3 is used to show that $T_{x, V}$ satisfies (6) for the case $x_i \geq a$. It indicates the relationships among the following four sets of state:

$$Z_1 = \{ (x - i e_i + (a - i + 1) e_j) : 0 \leq i \leq n \} ,$$

$$Z_2 = \{ (x - i e_i + (a - i) e_j) : 0 \leq i \leq n \} ,$$

$$Z_3 = \{ (x - (i - 1) e_i + (a - i + 1) e_j) : 0 \leq i \leq n \} ,$$

$$Z_4 = \{ (x - (i - 1) e_i + (a - i) e_j) : 0 \leq i \leq n \} .$$

To make our proof easier for readers to follow, we illustrate the relationships among the states in Lemma 3 in Figure 1. Here, $Q_n$, $R_i$, and $S_i$, $0 \leq i \leq n + 1$ are used to denote the possible state $x$.

Let $S_{n+i}$ denote $x - i e_i + (a - i) e_j$, $R_{n+i}$ denote $x - i e_i + (a - i + 1) e_j$, and $Q_{n+i}$ denote $x - (i - 1) e_i + (a - i + 1) e_j$. As an example, $S_n$ denotes $x + a e_j$, $R_n$ denotes $x + (a + 1) e_j$, and $Q_n$ denotes $x + (a - 1) e_j$. Thus, the four sets can be represented as $Z_1 = \{ R_0, R_1, ..., R_n \}$,

$Z_2 = \{ S_0, S_1, ..., S_n \}$, $Z_3 = \{ S_1, S_2, ..., S_{n+1} \}$ and $Z_4 = \{ Q_1, Q_2, ..., Q_{n+1} \}$, respectively.

-- Figure 1 here --

Define integers $i_1, i_2, i_3, i_4$, states $P_1, P_2, P_3, P_4$ and state set $M$ as:

$$i_1 = \min_{0 \leq i \leq n} \{ \arg \min v(R_i) \} , \quad i_2 = \min_{0 \leq i \leq n} \{ \arg \min v(S_i) \} , \quad i_3 = \min_{1 \leq i \leq n+1} \{ \arg \min v(S_i) \} ,$$

$$i_4 = \min_{1 \leq i \leq n+1} \{ \arg \min v(Q_i) \} , \quad P_1 = R_{i_1} , \quad P_2 = S_{i_2} , \quad P_3 = S_{i_3} , \quad P_4 = Q_{i_4} , \quad \text{and} \quad M = \{ P_1, P_2, P_3, P_4 \} .$$

It is easy to verify that $i_1 = n - i^*_1$, $i_2 = n - i^*_2$, $i_3 = n + 1 - i^*_3$ and $i_4 = n + 1 - i^*_4$ hold. Thus, Lemma 3 can be presented graphically as follows:

(1) If $P_2 = S_0$, $P_3 = S_1$, then $P_1 = R_0$ and $P_4 = Q_1$. 

\[(11)\]
If $P_2 = S_n$, $P_3 = S_{n+1}$, then $P_1 = R_n$ and $P_4 = Q_{n+1}$ \hfill (12)

(2) If $P_2 \neq S_0$ and $P_3 \neq S_{n+1}$, then $P_2 = P_3$. Furthermore,

If $P_2 = P_3 = S_k$ \((1 \leq k \leq n)\), then $P_1 = R_k$ and $P_4 = Q_k$ \hfill (13)

**Remark.** To show that $T_{ni}v$ satisfies (6) for the case $0 \leq x_i < a$, we need revise the four sets as:

$$Z_1 = \{R_0, R_1, ..., R_n\}, \ Z_2 = \{S_0, S_1, ..., S_n\}, \ Z_3 = \{S_0, S_1, ..., S_{n+1}\} \text{ and } Z_4 = \{Q_0, Q_1, ..., Q_{n+1}\}.$$

We have

(1) If $P_2 = S_n$, $P_3 = S_{n+1}$, then $P_1 = R_n$ and $P_4 = Q_{n+1}$

(2) If $P_2 = P_3 = S_k$ \((0 \leq k \leq n)\), then $P_1 = R_k$ and $P_4 = Q_k$

The proof for the case $0 < x_i < a$ is similar and thus omitted.

**Proof of Lemma 3**

It is equivalent to verify (11), (12) and (13). For (11) and (12), we only prove (11), since the proof for (12) is similar. As shown in Figure 1, if $P_2 = S_0$, $P_3 = S_1$ and $P_4 \neq R_0$, suppose $P_4 = R_k$,

\(1 \leq k \leq n\), by the definition of $P_1$ and Lemma 2,

$$v(R_0) \geq v(R_1) \geq ... \geq v(R_{k-1}) > v(R_k),$$

and because $v(S_0) \leq v(S_1) \leq ... \leq v(S_k)$, we have

$$v(S_0) + v(R_k) < v(R_0) + v(S_k). \quad (14)$$

On the other hand, by (8), we have $v(S_{i-1}) + v(R_i) \geq v(R_{i+1}) + v(S_i)$ for $1 \leq i \leq k$. Therefore,

$$v(S_0) + v(R_k) \geq v(R_0) + v(S_k)$$

which contradicts with (14). Similarly, $P_4 = Q_i$ can be obtained by (7).

For (13), if $P_2 \neq S_0$ and $P_3 \neq S_{n+1}$, then $P_2 = P_3$ can be obtained directly by the definition of $P_2$ and $P_3$. Now suppose $P_2 = P_3 = S_k$, \(1 \leq k \leq n\). By (8) and the definition of $P_2$, $P_1$ can only be some $R_j$, $j \geq k$. However, it is easy to verify that the case $j > k$ can be precluded since it contradicts with (8). Similarly, we have $P_4 = Q_k$. \(\blacksquare\)
The following properties in Lemma 4 and Lemma 5 are needed to show that \( T_{i_1}v \) satisfies (7) and (8), respectively.

**Lemma 4.** Suppose \( n = a \leq x_i \). For any demand class \( l, \ 2 \leq l \leq N \), define

\[
i_1^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 2)e_1 + (a - i)e_i) \},
\]

\[
i_2^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_1 + (a - i)e_i) \},
\]

\[
i_3^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_1 + (a - i)e_i) \},
\]

\[
i_4^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_1 + (a - i - 1)e_i) \}.
\]

Then,

1. If \( i_1^* = i_4^* = n \), then \( i_1^* = i_2^* = n \).

   If \( i_1^* = i_2^* = 0 \), then \( i_1^* = i_4^* = 0 \).

2. If \( i_1^* = i_2^* = n \), then either \( i_3^* = i_4^* = n \), or \( i_3^* = n \) and \( i_4^* = n - 1 \).

   If \( i_3^* = i_4^* = 0 \), then either \( i_1^* = i_2^* = 0 \), or \( i_1^* = 1 \) and \( i_2^* = 0 \).

We put graphical interpretation as follows: in Figure 2, we use \( R_i \) and \( S_i \), \( 0 \leq i \leq n + 1 \) to denote the possible \( x \), where

\[
T_{i_1}v(x + 2e_1) = \min_{0 \leq k \leq n} v(x - (k - 2)e_1 + (a - k)e_i) = \min_{1 \leq l \leq n + 1} v(S_l)
\]

\[
T_{i_1}v(x + e_1 - e_1) = \min_{0 \leq k \leq n} v(x - (k - 1)e_1 + (a - k - 1)e_i) = \min_{0 \leq l \leq n} v(S_l)
\]

\[
T_{i_1}v(x + e_i) = \min_{0 \leq k \leq n} v(x - (k - 1)e_1 + (a - k)e_i) = \min_{1 \leq l \leq n + 1} v(R_l)
\]

\[
T_{i_1}v(x - e_i) = \min_{0 \leq k \leq n} v(x - ke_1 + (a - k - 1)e_i) = \min_{0 \leq l \leq n} v(R_l)
\]

-- Figure 2 here --

Define integers \( i_1, i_2, i_3, i_4 \), states \( P_1, P_2, P_3, P_4 \) and state set \( M \) as
\[ i_1 = \min \{ \arg \min_{S_{i+1}} v(S_i) \}, \quad i_2 = \min \{ \arg \min_{S_i} v(S_i) \}, \quad i_3 = \min \{ \arg \min_{S_{i+1}} v(R_i) \}, \]

\[ i_4 = \min \{ \arg \min_{S_i} v(R_i) \}, \quad P_1 = S_{i}, \quad P_2 = S_{i}, \quad P_3 = R_{i}, \quad P_4 = R_{i} \quad \text{and} \quad M = \{ P_1, P_2, P_3, P_4 \} \]

Then, Lemma 4 is equivalent to the follows:

(1) If \( P_3 = R_1 \) and \( P_4 = R_0 \), then \( P_1 = S_1 \) and \( P_2 = S_0 \).

(2) If \( P_3 = S_1 \) and \( P_2 = S_0 \), then \( P_3 = R_1 \) and \( P_4 = R_0 \), or \( P_1 = P_2 = R_1 \).

(3) If \( P_3 = R_{n+1} \) and \( P_4 = R_n \), then \( P_1 = S_{n+1} \) and \( P_2 = S_n \), or \( P_1 = P_2 = S_n \).

\textbf{Remark.} To show that \( T_{la} v \) satisfies (7) for the case \( 0 \leq \alpha < a \), we need make the following revision:

\[ T_{la} v(x + 2e_i) = \min_{-1 \leq e_1 \leq 1} v(S_i), \quad T_{la} v(x + e_1 - e_i) = \min_{0 \leq e_1 \leq 1} v(S_i), \quad T_{la} v(x + e_i) = \min_{-1 \leq e_1 \leq 1} v(R_i) \]

and \( T_{la} v(x - e_i) = \min_{0 \leq e_1 \leq 1} v(R_i) \). Similarly, we can prove

(1) If \( P_1 = S_{n+1} \), \( P_2 = S_n \), then \( P_3 = R_{n+1} \) and \( P_4 = R_n \).

(2) If \( P_3 = R_{n+1} \), \( P_4 = R_n \), then \( P_1 = S_{n+1} \) and \( P_2 = S_n \), or \( P_1 = P_2 = S_n \).

(3) If \( P_1 = P_2 = S_k \) (\( -1 \leq k \leq n-1 \)), then \( P_3 = P_4 = R_{k+1} \).

\textbf{Proof of Lemma 4}

We only need to verify (15) – (18). As shown in Figure 2, for (15), if \( P_2 \neq S_0 \), suppose \( P_2 = S_k \), \( 1 \leq k \leq n \). By the definition of \( P_2 \) and Lemma 2,

\[ v(S_0) \geq v(S_1) \geq ... \geq v(S_{k-1}) > v(S_k) \]

and because \( v(R_0) \leq v(R_1) \leq ... \leq v(R_k) \), we have

\[ v(R_0) + v(S_k) < v(S_0) + v(R_k) \]

On the other hand, by (7),

\[ v(R_0) + v(S_1) \geq v(S_0) + v(R_1) \]

\[ v(R_1) + v(S_2) \geq v(S_1) + v(R_2) \]

\[ ... \]

\[ v(R_{k-1}) + v(S_k) \geq v(S_{k-1}) + v(R_k) \]
Therefore, \( v(R_0) + v(S_k) \geq v(S_0) + v(R_k) \), which contradicts the above derived result \( v(R_0) + v(S_k) < v(S_0) + v(R_k) \). Similarly, we have (16).

For (17), suppose \( P_k = R_k \), \( 2 \leq k \leq n \). Then we have

\[
v(R_1) \geq v(R_2) \geq \ldots \geq v(R_{k-1}) > v(R_k)
\]

Because

\[
v(S_0) \leq v(S_1) \leq \ldots \leq v(S_{k-1})
\]

Hence

\[
v(S_0) + v(R_k) < v(R_1) + v(S_{k-1})
\]

On the other hand, by (8)

\[
v(S_0) + v(R_2) \geq v(R_1) + v(S_1)
v(S_1) + v(R_3) \geq v(R_2) + v(S_2)
\]

\[
\vdots
\]

\[
v(S_k) + v(R_k) \geq v(R_{k-1}) + v(S_{k-1})
\]

Therefore

\[
v(S_0) + v(R_k) \geq v(R_1) + v(S_{k-1})
\]

which contradicts the above \( v(S_0) + v(R_k) < v(R_1) + v(S_{k-1}) \). Similarly, we have (18). ■

**Lemma 5** Suppose \( n = a \leq x_i \). For any demand class \( l \), \( 2 \leq l \leq N \), define

\[
i_1^* = \max \{ \arg \min_{0 \leq i \leq n} v(x-(i-1)e_i + (a-i)e_j) \},
\]

\[
i_2^* = \max \{ \arg \min_{0 \leq i \leq n} v(x-ie_i + (a-i-1)e_j) \},
\]

\[
i_3^* = \max \{ \arg \min_{0 \leq i \leq n} v(x-(i-1)e_i + (a-i+1)e_j) \},
\]

\[
i_4^* = \max \{ \arg \min_{0 \leq i \leq n} v(x-ie_i + (a-i)e_j) \}.
\]

Then,

1. If \( i_1^* = i_4^* = 0 \) or \( i_1^* = i_4^* = n \), then \( i_1^* = i_2^* = i_3^* = i_4^* \).

   If \( i_1^* = i_2^* = 0 \) or \( i_1^* = i_2^* = n \), then \( i_1^* = i_2^* = i_3^* = i_4^* \).
(2) If \( \hat{i}^* = \hat{i}^* + 1 \), then \( \hat{i}_3 = i_1^* \) and \( \hat{i}_4 = i_2^* \).

If \( \hat{i}_j = i_j^* + 1 \), then \( i_1^* = \hat{i}_3 \) and \( i_2^* = \hat{i}_4 \).

Graphical interpretation is as follow: in Figure 3, we use \( R_i \) and \( S_i \), \( 0 \leq i \leq n + 1 \) to denote the possible \( x \), where

\[
T_\nu v(x + e_i) = \min_{0 \leq i \leq S(x \wedge a)} v(x - (k - 1)e_i + (a - k)e_i) = \min_{1 \leq S \leq n + 1} v(S_i)
\]

\[
T_\nu v(x - e_i) = \min_{0 \leq i \leq S(x \wedge a)} v(x - ke_i + (a - k - 1)e_i) = \min_{0 \leq S \leq n} v(S_i)
\]

\[
T_\nu v(x + e_i + e_i) = \min_{0 \leq i \leq S(x \wedge a)} v(x - (k - 1)e_i + (a - k + 1)e_i) = \min_{1 \leq S \leq n + 1} v(R_i)
\]

\[
T_\nu v(x) = \min_{0 \leq i \leq S(x \wedge a)} v(x - ke_i + (a - k)e_i) = \min_{0 \leq S \leq n} v(R_i)
\]

Define integers \( i_1, i_2, i_3, i_4 \), states \( P_1, P_2, P_3, P_4 \) and state set \( M \) as

\[
i_1 = \min \{ \arg \min_{1 \leq S \leq n + 1} v(S_i) \}, \quad i_2 = \min \{ \arg \min_{0 \leq S \leq n} v(S_i) \}, \quad i_3 = \min \{ \arg \min_{1 \leq S \leq n + 1} v(R_i) \}, \quad i_4 = \min \{ \arg \min_{0 \leq S \leq n} v(R_i) \},
\]

\[
P_1 = S_1, \quad P_2 = S_2, \quad P_3 = R_n, \quad P_4 = R_4, \quad M = \{ P_1, P_2, P_3, P_4 \}.
\]

Then Lemma 5 is equivalent to the follows:

\[
(1) \quad P_3 = R_n \quad \text{and} \quad P_4 = R_0 \Leftrightarrow P_1 = S_1 \quad \text{and} \quad P_2 = S_0 \quad \text{(19)}
\]

\[
P_3 = R_{n+1} \quad \text{and} \quad P_4 = R_{n} \Leftrightarrow P_1 = S_{n+1} \quad \text{and} \quad P_2 = S_{n} \quad \text{(20)}
\]

\[
(2) \quad P_1 = P_2 = S_k \Leftrightarrow P_3 = P_4 = R_k, \quad 0 < k < n + 1 \quad \text{(21)}
\]

**Remark.** To show that \( T_\nu v \) satisfies (8) for the case \( 0 \leq x_i < a \), we need make the following revision: \( T_\nu v(x + e_i) = \min_{0 \leq S \leq n + 1} v(S_i) \), \( T_\nu v(x - e_i) = \min_{0 \leq S \leq n} v(S_i) \), \( T_\nu v(x + e_i + e_i) = \min_{1 \leq S \leq n + 1} v(R_i) \)

and \( T_\nu v(x) = \min_{0 \leq S \leq n} v(R_i) \). Similarly, we can prove

\[
(1) \quad P_3 = R_{n+1} \quad \text{and} \quad P_4 = R_{n} \Leftrightarrow P_1 = S_{n+1} \quad \text{and} \quad P_2 = S_{n}.
\]

\[
(2) \quad P_1 = P_2 = S_k \Leftrightarrow P_3 = P_4 = R_k, \quad 0 \leq k < n + 1.
\]
Proof of Lemma 5

As shown in Figure 3: For (19) and (20), we only verify (19).

\( \Rightarrow \): First, if \( P_1 = S_1 \), then from (8) we have \( P_2 = S_0 \). We proceed to prove \( P_1 = S_1 \). Otherwise, if \( P_1 = S_k \), \( 2 \leq k \leq n \), then from Lemma 2, we have \( \nu(S_0) \geq \nu(S_i) \geq \ldots \geq \nu(S_{k-1}) > \nu(S_k) \). Since \( \nu(R_0) \leq \nu(R_i) \leq \ldots \leq \nu(R_k) \), we have \( \nu(R_{k-1}) + \nu(S_k) < \nu(R_k) + \nu(S_{k-1}) \), which contradicts (7). If \( P_1 = S_0 \), it implies \( \nu(S_0) < \nu(S_i) \) and thus \( \nu(R_0) < \nu(R_i) \), which is a contradiction to \( P_3 = R_i \).

\( \Leftarrow \): First, if \( P_3 = R_i \), then from (8), we have \( P_2 = R_0 \). We proceed to prove \( P_3 = R_i \). Otherwise, if \( P_3 = R_k \), \( 2 \leq k \leq n \), similarly, we have \( \nu(S_0) + \nu(R_k) < \nu(S_k) + \nu(R_0) \), which contradicts (8). If \( P_3 = R_0 \), it implies \( P_1 = S_0 \), thus a contradiction follows. Further, (21) can be easily obtained from (8).

Proof of \( T_{la}v \in V \ (2 \leq l \leq N) \)

For (3) and (4), since \( x_i < 0 \), \( x_i + 1 \leq 0 \), we have \( T_{la}v(x) = v(x + ae_i) \),

\[
T_{la}v(x + e_i) = v(x + e_i + ae_i), \quad \text{and} \quad T_{la}v(x-e_i) = v(x + (a-1)e_i).
\]

Since \( v \in V \), we have

\[
T_{la}v(x + e_i) = v(x + e_i + ae_i) \leq v(x + ae_i) = T_{la}v(x)
\]

\[
T_{la}v(x + e_i) = v(x + e_i + ae_i) \leq v(x + (a-1)e_i) = T_{la}v(x-e_i)
\]

For (5), consider two cases as follow:

(i) \( x_i \leq 0 \). We have \( T_{la}v(x) = v(x + ae_i) \) and \( T_{la}v(x-e_i) = v(x + (a-1)e_i) \). From (5), we have \( v(x + (a-1)e_i) \leq v(x + ae_i) \). Thus \( T_{la}v(x-e_i) \leq T_{la}v(x) \).

(ii) \( x_i > 0 \). From (6), we have \( v(x - ke_i + (a-k)1e_i) \leq v(x - ke_i + (a-k)e_i) \) for \( k = 0, \ldots, (x \wedge a) \). Thus,

\[
T_{la}v(x-e_i) = \min_{0 \leq k \leq (x \wedge a)} v(x - ke_i + (a-k)1e_i) \leq \min_{0 \leq k \leq (x \wedge a)} v(x - ke_i + (a-k)e_i) = T_{la}v(x).
\]

For (6), we consider three cases as follow:
(i) \( x_i \geq a \). Following the notations in the proof of Lemma 3 (see Figure 1), we enumerate all the cases of \( M = \{P_1, P_2, P_3, P_4\} \):

1. \( P_1 = R_0, \ P_2 = S_0, \ P_3 = S_4, \ P_4 = Q_4 \)
2. \( P_1 = R_n, \ P_2 = S_n, \ P_3 = S_{n+1}, \ P_4 = Q_{n+1} \)
3. \( P_2 = P_3 = S_k, \ P_1 = R_k, \ P_4 = Q_k, \ 1 \leq k \leq n \)

For each case, by the convexity of \( v \) and (6) – (8), it is straightforward to show that \( T_{\nu}v \) satisfies (6).

(ii) \( 0 \leq x_i < a \), we enumerate all the cases

1. \( P_1 = R_n, \ P_2 = S_n, \ P_3 = S_{n+1} \) and \( P_4 = Q_{n+1} \)
2. \( P_2 = P_3 = S_k, \ P_1 = R_k \) and \( P_4 = Q_k, \ 0 \leq k \leq n \)

Similarly, we have that \( T_{\nu}v \) satisfies (6).

(iii) \( x_i < 0 \). We have

\[
T_{\nu}v(x) = v(x + ae_i)
\]

\[
T_{\nu}v(x + e_i) = v(x + (a + 1)e_i)
\]

\[
T_{\nu}v(x + e_i) = \begin{cases} 
\min[v(x + e_i + ae_i), v(x + (a - 1)e_i)] & \text{if } x_i = 0 \\
v(x + e_i + ae_i) & \text{if } x_i < 0 
\end{cases}
\]

\[
T_{\nu}v(x + e_i + e_i) = \begin{cases} 
\min[v(x + e_i + (a + 1)e_i), v(x + ae_i)] & \text{if } x_i = 0 \\
v(x + e_i + (a + 1)e_i) & \text{if } x_i < 0 
\end{cases}
\]

If \( T_{\nu}v(x + e_i) \neq v(x + (a - 1)e_i) \), then \( x_i = 0 \) and

\[
T_{\nu}v(x + e_i + e_i) + T_{\nu}v(x) \leq v(x + ae_i) + v(x + ae_i) \\
\leq v(x + (a + 1)e_i) + v(x + (a - 1)e_i) = T_{\nu}v(x + e_i) + T_{\nu}v(x + e_i)
\]

where the second inequality follows from the convexity of \( v \).

If \( T_{\nu}v(x + e_i) = v(x + e_i + ae_i) \), then

\[
T_{\nu}v(x + e_i + e_i) + T_{\nu}v(x) \leq v(x + e_i + (a + 1)e_i) + v(x + ae_i) \\
\leq v(x + e_i + ae_i) + v(x + (a + 1)e_i) = T_{\nu}v(x + e_i) + T_{\nu}v(x + e_i)
\]

where the second inequality follows from the fact that \( v \) satisfies (6).
For (7) we consider three cases as follow:

(i) \( x_i > a \). Following the notations in the proof of Lemma 4 (see Figure 2), we enumerate all the cases of \( M = \{P_1, P_2, P_3, P_4\} \): 

1. \( P_1 = S_1, \ P_2 = S_0, \ P_3 = R_1, \ P_4 = R_0 \); 
2. \( P_1 = S_{n+1}, \ P_2 = S_n, \ P_3 = R_{n+1}, \ P_4 = R_n \); 
3. \( P_1 = S_1, \ P_2 = S_0, \ P_3 = P_4 = R_1 \); 
4. \( P_1 = P_2 = S_n, P_3 = R_{n+1}, \ P_4 = R_n \); 
5. \( P_1 = P_2 = S_k, \ P_3 = P_4 = R_{k+1}, \ 1 \leq k \leq n \).

For each case, it is straightforward to show that \( T_{lu}v \) satisfies (7).

(ii) \( 0 \leq x_i < a \). we enumerate all the cases

1. \( P_1 = S_{n+1}, \ P_2 = S_n, \ P_3 = R_{n+1} \) and \( P_4 = R_n \).
2. \( P_1 = P_2 = S_n, \ P_3 = R_{n+1} \) and \( P_4 = R_n \).
3. \( P_1 = P_2 = S_k, \ P_3 = P_4 = R_{k+1}, \ -1 \leq k \leq n - 1 \).

For each case, we have that \( T_{lu}v \) satisfies (7).

(iii) \( x_i < 0 \). We have

\[
T_{lu}v(x-e_i) = v(x + (a-1)e_i) \]

\[
T_{lu}v(x + 2e_i) = \begin{cases} 
\min[v(x + 2e_i + ae_i), v(x + e_i + (a-1)e_i), v(x + (a-2)e_i)] & \text{if } x_i = 0 \\
\min[v(x + 2e_i + ae_i), v(x + e_i + (a-1)e_i)] & \text{if } x_i = -1 \\
v(x + 2e_i + ae_i) & \text{if } x_i \leq -2
\end{cases}
\]

If \( T_{lu}v(x + 2e_i) = v(x + 2e_i + ae_i) \), then

\[
T_{lu}v(x + e_i - e_j) + T_{lu}v(x + e_j) \leq v(x + e_i + (a-1)e_i) + v(x + e_j + ae_j) \\
\leq v(x + 2e_i + ae_i) + v(x + (a-1)e_i) = T_{lu}v(x + 2e_i) + T_{lu}v(x - e_j)
\]

where the first inequality follows from the definition of \( T_{lu} \), and the second one is from the fact that \( v \) satisfies (7).

If \( T_{lu}v(x + 2e_i) = v(x + e_i + (a-1)e_i) \) and \( x_i = -1 \), then

\[
T_{lu}v(x + e_i - e_j) + T_{lu}v(x + e_j) = v(x + e_i + (a-1)e_i) + v(x + e_i + ae_j) \\
\leq v(x + e_i + (a-1)e_j) + v(x + (a-1)e_j) = T_{lu}v(x + 2e_i) + T_{lu}v(x - e_j)
\]

where the first inequality follows from the fact that \( v \) satisfies (4).
If $x_i = 0$, then

$$T_{\text{in}} v(x + e_i - e_i) = \min[v(x + e_i + (a-1)e_i), v(x + (a-2)e_i)]$$

$$T_{\text{in}} v(x + e_i) = \min[v(x + e_i + ae_i), v(x + (a-1)e_i)]$$

Hence, if $T_{\text{in}} v(x + 2e_i) = v(x + e_i + (a-1)e_i)$, then

$$T_{\text{in}} v(x + e_i - e_i) + T_{\text{in}} v(x + e_i) \leq v(x + e_i + (a-1)e_i) + v(x + (a-1)e_i)$$
$$= T_{\text{in}} v(x + 2e_i) + T_{\text{in}} v(x - e_i)$$

If $T_{\text{in}} v(x + 2e_i) = v(x + (a-2)e_i)$, we have

$$T_{\text{in}} v(x + e_i - e_i) + T_{\text{in}} v(x + e_i) \leq v(x + (a-2)e_i) + v(x + (a-1)e_i)$$
$$= T_{\text{in}} v(x + 2e_i) + T_{\text{in}} v(x - e_i)$$

For (8), we consider three cases as follow:

(i) $x_i \geq a$. From Lemma 5, it is straightforward to show that $T_{\text{in}} v$ satisfies (8).

(ii) $0 \leq x_i < a$. It is straightforward that $T_{\text{in}} v$ satisfies (8).

(iii) $x_i < 0$. We have

$$T_{\text{in}} v(x) = v(x + ae_i)$$
$$T_{\text{in}} v(x - e_i) = v(x + (a-1)e_i)$$
$$T_{\text{in}} v(x + e_i + e_i) = \begin{cases} \min[v(x + ae_i), v(x + e_i + (a+1)e_i)] & \text{if } x_i = 0 \\ v(x + e_i + (a+1)e_i) & \text{if } x_i \leq -1 \end{cases}$$

If $T_{\text{in}} v(x + e_i + e_i) = v(x + e_i + (a+1)e_i)$, then

$$T_{\text{in}} v(x + e_i) + T_{\text{in}} v(x) \leq v(x + e_i + ae_i) + v(x + ae_i)$$
$$\leq v(x + e_i + (a+1)e_i) + v(x + (a-1)e_i) = T_{\text{in}} v(x + e_i + e_i) + T_{\text{in}} v(x - e_i)$$

where the first inequality follows from the definition of $T_{\text{in}}$, and the second one is from the fact that $v$ satisfies (7).

If $T_{\text{in}} v(x + e_i + e_i) = v(x + ae_i)$, then $x_i = 0$, and

$$T_{\text{in}} v(x + e_i) + T_{\text{in}} v(x) \leq v(x + (a-1)e_i) + v(x + ae_i) = T_{\text{in}} v(x - e_i) + T_{\text{in}} v(x + e_i + e_i)$$

where the inequality follows from the fact that

$$T_{\text{in}} v(x + e_i) = \min[v(x + (a-1)e_i), v(x + e_i + ae_i)]$$

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Proof of $T_0v \in V$

For (3), we need to show $T_0v(x + e_i) \leq T_0v(x)$. This is obvious, since for the optimal policy $\pi$ at $x$ (note the production quantity is positive), adopting $\pi$ except for one less unit production at $x + e_i$ results in the same cost.

For (4), suppose $(k_1, \ldots, k_N)$ is the optimal policy for $x - e_i$. We can construct a suitable policy $(k_1', \ldots, k_N')$ for $x + e_i$ (not necessary optimal): Others same as $(k_1, \ldots, k_N)$, let $k_i' = k_i + 1$. This leads to the same cost at the two states.

For (5), suppose $(k_1, \ldots, k_N)$ is the optimal policy for $x$. Consider the following cases:

(i) No production at $x$. It implies also no production at $x - e_i$. The result is immediate since no allocation.

(ii) Produce and $k_i < x_i$. We can construct a suitable policy $(k_1', \ldots, k_N')$ for $x - e_i$ (not necessary optimal): $k_i' = k_i - 1$ and others same as $(k_1, \ldots, k_N)$. From (5), this leads to a higher (or equal) cost at $x$.

(iii) Produce and $k_i = x_i$. There are two possibilities: (a) $T_0v(x - e_i) = v(x_i'e_i)$ and $T_0v(x) = v((x_i' - 1)e_i)$ for some $x_i' \in Z$. Notice there must be $x_i' \leq z_{N+1}$, from (9), we have $T_0v(x - e_i) \leq T_0v(x)$ . (b) $T_0v(x - e_i) = v(x' - e_j)$ and $T_0v(x) = v(x')$ for some $x' = (x'_1, 0, \ldots, 0, x'_j, \ldots, x'_n)$, where $l < j \leq N$ and $x'_j > 0$. The result follows from (5).

For (6), denote $P = x = (x_1, 0, \ldots, 0, x_j, x_{j+1}, \ldots, x_N)$, $P_i = x + e_i$, $P_j = x + e_j$ and $P_{ij} = x + e_i + e_j$. We need show $T_0v(P_{ij}) + T_0v(P) \leq T_0v(P_i) + T_0v(P_j)$. We only discuss the case $0 < x_i + 1 \leq z_i$, because any state with $x_i + 1 > z_i$ is transient. Suppose $T_0v(P) = v(P')$, i.e., after the optimal decision, $P$ moves to $P' = x'$. Similarly, $T_0v(P_i) = v(P_i')$, $T_0v(P_j) = v(P_j')$ and $T_0v(P_{ij}) = v(P_{ij}')$. Consider the following cases:
\( x_i + 1 + M \leq z_j \). Then we have \( P' = x + Me_1, \quad P'_1 = x + e_1 + Me_1, \quad P'_l = x + e_1 + e + Me_1 \). The result follows directly from (6).

(ii) \( z_j < x_i + 1 + M \leq z_i + x_i \). It is easy to verify that

\[
P' = P'_l = x + (z_i - x_i) e_1 - (M - z_i + x_i) e_j,
\]

\[
P'_1 = (x + e_1) + (z_i - x - 1) e_1 - (M - z_i + x_i + 1) e_j = x + (z_i - x) e_1 - (M - z_i + x_i + 1) e_j,
\]

\[
P'_l = (x + e_1) + (z_i - x) e_1 - (M - z_i + x_i) e_j = x + (z_i - x) e_1 - (M - z_i + x_i - 1) e_j.
\]

The result follows directly from (9).

(iii) \( z_j + x_j + 1 = x_i + 1 + M \). It is easy to verify that \( P' = P'_l = x + (z_i - x_i) e_1 - x_i e_j \), \( P'_1 = x + (z_i - x_i + 1) e_1 - x_i e_j \), and \( P'_l = x + (z_i - x_i) e_1 - (x_i - 1) e_j \).

Let \( P'' = x + (z_i - x_i + 1) e_1 - (x_i - 1) e_j \). Obviously, \( v(P'') \geq v(P') \). We have

\[
v(P') + v(P'_l') \leq v(P'') + v(P'_l') \leq v(P'_1) + v(P'_l'),
\]

where the second inequality is from (6).

(iv) \( z_j + x_j + 1 < x_i + 1 + M \leq z_N + 1 + \sum_{i=1}^{N} x_i \). It can be shown that for all the four states, it is optimal to produce \( M \) units, satisfy all the class \( l \) backorder: First, \( P \) moves to \( P'' = x + (M - x_i) e_1 - x_i e_j \), \( P_1 \) moves to \( P''_1 = x + (1 + M - x_i) e_1 - x_i e_j \), \( P_1 \) moves to \( P''_l = x + (M - x_i - 1) e_1 - x_i e_j \), \( P_{l} \) moves to \( P''_{l} = x + (M - x_i) e_1 - x_i e_j \), and then it need to consider allocating production to the backorders from less valuable classes. From (9), we can show that \( v(P'') + v(P''_{l}) \leq v(P''_1) + v(P''_{l}) \) by following similar analysis in Lemma 3 - 5.

(v) \( x_i + 1 + M = z_{N+1} + \sum_{i=1}^{N} x_i + 1 \). It can be shown that it is optimal to produce \( M \) units and satisfy all backorders at \( P \) and \( P_{l} \), to produce \( M - 1 \) units and satisfy all backorders at \( P_1 \), and to produce \( M \) units and leave one unit of backorder unsatisfied at \( P_{l} \). Finally, \( P' = P'_l = P'_l = z_{N+1} e_1, \quad P'_l = z_{N+1} e_1 + e_j \) for some \( j > l \). The result is obvious.
(vi) \( x_i + M \geq z_{N+i} + \sum_{j=1}^{N} x_j + 1 \). It can be shown that \( P' = P'_{il} = P'_1 = P'_i = z_{N+i} e_i \).

For (7), denote \( P = x = (x_i,0,...,0,x_i,x_{i+1},...,x_N) \), \( P_1 = x + e_i \), \( P_{il} = x + e_i + e_i \) and \( P_{2l} = x + 2e_i + e_i \). We need show \( T_0 v(P_{il}) + T_0 v(P_1) \leq T_0 v(P) + T_0 v(P_{2l}) \). We will discuss the case \( 0 < x_i + 2 \leq z_i \), as the case \( z_i < 2 \) or \( x_i + 2 > z_i \) is trivial. Suppose \( T_0 v(P) = v(P') \), \( T_0 v(P_1) = v(P'_1) \), \( T_0 v(P_{il}) = v(P'_{il}) \) and \( T_0 v(P_{2l}) = v(P'_{2l}) \). Consider the following cases:

(i) \( x_i + 2 + M \leq z_i \). Then we have \( P' = x + Me_i \), \( P'_1 = x + e_i + Me_i \), \( P'_i = x + e_i + (M+1)e_i \), \( P'_{2l} = x + e_i + (M+2)e_i \). The result follows directly from (7).

(ii) \( x_i + 2 + M = z_i + 1 \). It is easy to verify that \( P'_1 = P'_{2l} = x + (M+1)e_i \), \( P' = x + Me_i \), \( P'_i = x + e_i + (M+1)e_i \). Since \( x_i + M < z_i \), we have \( v(P') \geq v(P'_{il}) \), and thus the result holds.

(iii) \( x_i + 2 + M > z_i + 1 \). It can be shown that \( P'_1 = P'_{2l} \) and \( P' = P'_{il} \).

For (8), denote \( P = x = (x_i,0,...,0,x_i,x_{i+1},...,x_N) \), \( P_1 = x + e_i \), \( P_{il} = x + e_i + e_i \) and \( P_{1,2l} = x + e_i + 2e_i \). We need show \( T_0 v(P_{il}) + T_0 v(P_1) \leq T_0 v(P) + T_0 v(P_{1,2l}) \). We will assume \( 0 < x_i + 1 \leq z_i \), as the case \( z_i < 1 \) or \( x_i + 1 > z_i \) is trivial. Suppose \( T_0 v(P) = v(P') \), \( T_0 v(P_1) = v(P'_1) \), \( T_0 v(P_{il}) = v(P'_{il}) \) and \( T_0 v(P_{1,2l}) = v(P'_{1,2l}) \). Consider the following cases:

(i) \( x_i + 1 + M \leq z_i \). Then we have \( P' = x + Me_i \), \( P'_1 = x + e_i + Me_i \), \( P'_i = x + e_i + (M+1)e_i \), \( P'_{1,2l} = x + (M+1)e_i + 2e_i \). The result follows directly from (8).

(ii) \( x_i + 1 + M > z_i \). It is easy to verify that \( P'_1 = P'_{1,2l} \), \( P' = P'_{il} \). \( \blacksquare \)
Figures and Tables

Figure 1. Graphical representation for Lemma 3

Figure 2. Graphical representation for Lemma 4
Figure 3. Graphical representation for Lemma 5