Performance bounds of Energy Detection with Signal Uncertainty in Cognitive Radio Networks

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Abstract—The harmonic coexistence of secondary users (SUs) and primary users (PUs) in cognitive radio networks requires SUs to identify the idle spectrum bands. One common approach to achieve spectrum awareness is through spectrum sensing, which usually assumes known distributions of the received signals. However, due to the nature of wireless channels, such an assumption is often too strong to be realistic, and leads to unreliable detection performance in practical networks. In this paper, we study the sensing performance under distribution uncertainty, i.e., the actual distribution functions of the received signals are subject to ambiguity and not fully known. Firstly, we define a series of uncertainty models based on signals’ moment statistics in different spectrum conditions. Then we present mathematical formulations to study the detection performance corresponding to these uncertainty models. Moreover, in order to make use of the distribution information embedded in historical data, we extract a reference distribution from past channel observations, and define a new uncertainty model in terms of it. With this uncertainty model, we propose two iterative procedures to study the false alarm probability and detection probability, respectively. Numerical results show that the detection performance with a reference distribution is less conservative compared with that of the uncertainty models merely based on signal statistics.

Index Terms—Cognitive radio, spectrum sensing, distribution uncertainty, probabilistic distance measure.

I. INTRODUCTION

Spectrum awareness in cognitive radio network can be achieved either through the information exchange between PUs and SUs, or via SUs’ spectrum sensing and prediction. The first method requires the cooperation between PUs and SUs, which introduces extra signaling overhead and sometimes may be infeasible (e.g., [1], [2]). The second method does not rely on the information provided by PUs, but demands accurate sensing algorithms for SUs (e.g., [3], [4]). In this paper, we focus on the second method and study the sensing performance with uncertain distribution information about the received signals.

Energy-based spectrum sensing is generally preferred since it does not require pre-knowledge about PUs’ signals. The detection problem is then formulated as a hypothesis testing where the null hypothesis $H_0$ denotes an idle channel and the alternate hypothesis $H_1$ implies a busy channel. Based on the Neyman-Pearson criterion, an SU decides the presence of PUs by comparing a test statistic with a pre-designed decision threshold. The analysis of detection performance refers to the tradeoff study between detection probability (i.e., probability of correctly sensing a busy channel) and false alarm probability (i.e., error probability of sensing an idle channel as busy), which requires the knowledge of signal distributions in hypotheses $H_0$ and $H_1$. Most research works in literature assume that the signal distributions are known. For example, the authors in [5]–[8] derived analytical expressions for detection probability and false alarm probability given different channel models. However, due to the stochastic nature of wireless channels, it is usually very hard in practice to know precise information regarding primary signals’ probability distribution. For example, the received signals exhibit different distributions depending on whether there is line-of-sight between transceivers. The mobility of wireless nodes also affects the signal distributions significantly.

As uncertain factors commonly exist in practical networks, researchers have been trying to design robust sensing strategies that do not rely on parametric distributions, and are tolerable to signal fluctuations. The authors in [9] proposed a nonparametric correlation detector, which does not require closed-form distribution functions for noise and PUs’ signals. However, it still requires extra knowledge about the cyclic frequencies of PUs’ signals. Nonparametric detection methods were also proposed based on goodness of fit test, which evaluates whether a sequence of signal samples is drawn from some known distribution. The authors in [10], [11] constructed the test statistic by empirical distribution of the signal samples, assuming a known noise distribution in the goodness of fit test. Higher order signal statistics, such as skewness and kurtosis, were also employed in the goodness of fit test [12], [13] to improve the robustness against environment variations. Though nonparametric goodness of fit test assumes no pre-knowledge regarding PUs’ signal distribution, it needs to know the noise distribution in advance, and is usually difficult to study the detection performance analytically.

Compared with our previous works [14], [15], where we characterized the distribution uncertainty by moment constraints, in this work we present a comparative study on different uncertainty models for different scenarios, considering the uncertainties for both noise and PUs’ signals in spectrum sensing. Especially, we propose a new uncertainty model that is more practical and flexible in modeling signal fluctuations.
in real spectrum environment. The main contributions of this paper are as follows:

- **Uncertainty Models**: We define different uncertainty models that are applicable for different scenarios. In the first part, we consider the moment uncertainties that are based on moment statistics, e.g., sample mean and variance estimates of the received signals. In the second part, we exploit the distribution information embedded in the historical data, from which we extract a reference distribution by goodness of fit test. Then we propose a new uncertainty model in which the actual signal distribution is allowed to fluctuate around such reference distribution, and qualify their discrepancy in terms of a probabilistic distance measure.

- **Detection Bounds**: Based on different signal uncertainties, we study the sensing performance in terms of lower and upper detection bounds. For the moment uncertainties, we show that the detection bounds can be obtained from equivalent semi-definite programs (SDP) that are solvable by interior point algorithm, and we derive an analytical formula, which can help SUs to quickly assess the detection bounds, instead of solving a SDP. For the uncertainty with a reference distribution, we propose two iterative methods to obtain the false alarm probability and detection bounds, respectively.

The rest of this paper is organized as follows. Section II describes the spectrum sensing model and different signal uncertainties. In Section III, we study the detection performance with moment uncertainties, while in Section IV, we extend the study into the uncertainty model with a reference distribution. Section V gives some numerical results and Section VI concludes the whole paper.

II. SYSTEM MODEL

A. Single User Energy Detection

We consider multiple SUs and PUs sharing the same spectrum in a cognitive radio network. Assuming no information exchange between PUs and SUs, each SU needs to perform spectrum sensing independently. The objective of spectrum sensing is to detect the presence of primary signals on the primary channel, and we model this problem as a binary hypothesis testing. Specifically, if the received signal $\xi$ at an SU’s receiver exceeds a decision threshold $\lambda$, the SU’s detector will return 1 indicating the presence of PU on the sensing channel (i.e., hypothesis $H_1$), otherwise return 0 indicating an idle channel (i.e., hypothesis $H_0$). Thus the decision results can be viewed as drawn from a decision function $h(\xi, \lambda) = I(\xi > \lambda)$ where indicator $I(A)$ equals 1 if event $A$ is true and 0 otherwise. Given this decision function, we will study the tradeoff between detection probability $Q_d$ and false alarm probability $Q_f$. Let $f_0(\xi)$ be the distribution function of the received signal at an SU’s receiver when PUs are absent, then the false alarm probability is given by

$$Q_f = \mathbb{E}[h(\xi, \lambda)] = \int_{\xi \in S} h(\xi, \lambda) f_0(\xi) \, d\xi,$$

where $S$ denotes the set of all possible values of the received signal strength. When PUs are present, the received signal $\xi$ will exhibit a different distribution function $f_1(\xi)$, and we express the detection probability as follows:

$$Q_d = \mathbb{E}[h(\xi, \lambda)] = \int_{\xi \in S} h(\xi, \lambda) f_1(\xi) \, d\xi.$$

Then our task is to maximize the detection probability $Q_d$ subject to a false alarm probability requirement, i.e., $Q_f \leq \alpha$. From the above expressions, we note that the knowledge about two distribution functions $f_0(\xi)$ and $f_1(\xi)$ are crucial for an analytical study of the detection performance. However, in practice, it is usually impossible to obtain exact information about these distributions. That’s because, distribution $f_0(\xi)$ depends on environment noise and interference from other active users, which is usually time-varying. Besides, multiple PUs may have different transmission techniques and traffic patterns, therefore, the signal distribution $f_1(\xi)$ may also be highly dynamic in terms of the statistical information. In the following, we present several uncertainty models that quantify signal fluctuations in different scenarios.

B. Known Distribution with Uncertain Statistics (KDUS)

In some cases, the environment noise is relatively stable and the channel condition is well informed, or we are targeting at specific primary signals. Therefore, we are able to know or assume some parametric distribution function either for noise or PUs’ signals in advance [5]. However, we still need to estimate the distribution parameters (e.g., mean and variance) before we can fully characterize the signal distribution. Usually, parameter estimation is based on signal samples which bear limited information about the real signal distribution, thus estimation errors are inevitable, and we define the signal uncertainty in terms of these estimation errors:

$$U_d(\mu_0, \sigma_0, \gamma_\mu, \gamma_\sigma) = \left\{ f(\xi) \in \mathcal{G} \mid \frac{\mathbb{E}[f(\xi) - \mu_0]^2}{\mathbb{E}[f(\xi^2) - \mu_0^2]} \leq \frac{\gamma_\mu}{\gamma_\sigma^2}, \frac{\mathbb{E}[f(\xi^2) - \mu_0^2]}{\mathbb{E}[f(\xi^4) - \mu_0^4]} \leq \frac{\gamma_\sigma^2}{\gamma_\sigma^4} \right\}, \quad (1)$$

where $\mathcal{G}$ is a set of distribution functions that belong to the same family, e.g., Gaussian distributions. Due to the fluctuations of wireless channel, the sample mean $\mu_0$ and variance $\sigma_0^2$ are slightly different from the real distribution mean $\mu = \mathbb{E}[f(\xi)]$ and variance $\sigma^2 = \mathbb{E}[f(\xi^2) - \mu_0^2]$1. These two inequalities in (1) describe how likely the distribution mean $\mathbb{E}[f(\xi)]$ and the signal samples $\xi$ are close to the sample estimate $\mu_0$, respectively. Parameters $\gamma_\mu$ and $\gamma_\sigma$ regulate the uncertainty size and provide a way to evaluate the confidence in sample estimates $\mu_0$ and $\sigma_0^2$, respectively. A theoretical work in [16] demonstrated how to choose proper values for $\gamma_\mu$ and $\gamma_\sigma$ based on the sample size.

C. General Distribution with Exact Statistics (GDES)

If pre-knowledge about PUs is not available in practice, it is often difficult for the SU to know precisely the primary

1Here we replace $\mu$ by its estimation $\mu_0$ in the expression of distribution variance.
signal’s distribution function $f_1(\xi)$, as it is often time varying and experiences attenuation, shadowing, and multi-path fading before reaching the SU’s receiver. Instead, estimates of the signal statistics may provide a practical way to study the signal’s properties. If the signal detector is capable of sampling the primary channel for a long time, these sample estimates will be close to the real statistics with high confidence. In this paper, we consider PUs’ signal up to its second-order moment statistic and define the distribution uncertainty as follows:

$$U_c(\mu_0, \sigma_0) = \left\{ f(\xi) \in M \left| \begin{array}{l} \mathbb{E}_f[\xi] = \mu_0 \\
\mathbb{E}_f[(\xi - \mu)^2] = \sigma_0^2 \end{array} \right\} \right., \quad (2)$$

where $M$ denotes the set of all possible distribution functions, which only requires $\int_{\xi \in S} f(\xi) d\xi = 1$ and $f(\xi) \geq 0$ for all $\xi \in S$. In this model, we relax the distribution function $f(\xi)$ to be any form of distributions, while restrict the signal statistics by a series of equality constraints.

**D. General Distribution with Uncertain Statistics (GDUS)**

In a more general setting, the signal distribution is uncertain and the signal statistics are also subject to estimation errors due to limited observations and measurement noise. For example, the spectrum environment may have been changed while the detector still maintains outdated statistical information. Even though we may update the signal statistics by the most recent channel measurements, we still lack the confidence to entirely rely on these estimates. Instead, it is reasonable and safe to assume that these sample estimates are within small ranges of their real values, and we can express the distribution uncertainty as follows:

$$U_g(\mu_0, \sigma_0, \gamma_\mu, \gamma_\sigma) = \left\{ f(\xi) \in M \left| \begin{array}{l} (\mathbb{E}_f[\xi] - \mu_0)^2 \leq \gamma_\mu \sigma_0^2 \\
(\mathbb{E}_f[(\xi - \mu)^2] - \sigma_0^2) \leq \gamma_\sigma \sigma_0^2 \end{array} \right\} \right.. \quad (3)$$

Uncertainty (3) has the same inequality constraints as in set (1), their difference lies in the distribution sets $G$ and $M$. Note that $G$ represents a specific distribution family, while set $M$ does not pose any restrictions on its type. We also note that uncertainty (2) is a special case of set (3) when parameters $\gamma_\mu$ and $\gamma_\sigma$ are set to zeros.

**III. ROBUST DETECTION PERFORMANCE**

For the hypothesis testing problem, we first determine a decision threshold $\lambda$ according to the false alarm probability requirement, i.e., $Q_f \leq \alpha$ where $\alpha$ is the maximum tolerable false alarm probability. Since Gaussian distribution is a good approximation for noise signal in both theory and practice, we can model its fluctuations by set (1). Mathematically, the choice of decision threshold $\lambda$ should satisfy

$$Q_f^L(\lambda) = \max_{f_0(\xi)} \mathbb{E}_{f_0}[h(\xi, \lambda)] = \alpha, \quad (4)$$

where $f_0(\xi)$ is subject to set $U_d(\mu_{n,0}, \sigma_{n,0}, \gamma_{n,\mu}, \gamma_{n,\sigma})$. Parameters $\mu_{n,0}$ and $\sigma_{n,0}$ denote the sample estimates of noise signal’s distribution mean $\mu_n$ and variance $\sigma_n^2$, respectively, and their estimation errors are regulated by $\gamma_{n,\mu}$ and $\gamma_{n,\sigma}$, respectively. Function $Q_f^L(\lambda)$ denotes the worst-case false alarm probability with fixed decision threshold $\lambda$. When $f_0(\xi)$ follows a Gaussian distribution, we have $Q_f^L(\lambda) = 1 - \Phi\left(\frac{\lambda - \mu_n}{\sigma_n}\right)$ where $\Phi(\cdot)$ is the cumulative density function of standard Gaussian distribution, and the worst-case false alarm probability is given by

$$Q_f^L(\lambda) = 1 - \min_{\mu_n, \sigma_n} \Phi\left(\frac{\lambda - \mu_{n,0}}{\sigma_{n,0}}\right) \left(1 - \gamma_{n,\sigma}\right)^{-1}\left(1 - \alpha\right). \quad (5)$$

We note that the worst-case fluctuation occurs when the noise signal takes the largest mean $\mu_{n,0} + \sigma_{n,0}\sqrt{\gamma_{n,\mu}}$ and variance $\sigma_{n,0}^2(1 + \gamma_{n,\sigma})$ in its uncertainty set. Therefore, the robust decision threshold should be set as

$$\lambda = \mu_{n,0} + \sigma_{n,0}\sqrt{\gamma_{n,\mu}} + \sigma_{n,0}\sqrt{1 + \gamma_{n,\sigma}^2} \Phi^{-1}(1 - \alpha). \quad (6)$$

Given this decision threshold, we aim to analyze the sensing performance in terms of detection probability. However, deterministic study usually fails when distribution $f_1(\xi)$ is subject to uncertainties. Instead, we can study its lower and upper bounds, which can be used as important reference in real-world decision making. As we focus on the worst-case scenario, the following study intends to find a lower bound of detection probability. Similar approaches are also applicable for studying the upper bound of detection probability, which is omitted here due to space limit.

**A. LOWER BOUND OF DETECTION PROBABILITY**

Considering the uncertainty in (1) for primary signal, the lower detection bound can be processed in the same way as for false alarm probability. For example, if $f_1(\xi)$ is also a Gaussian distribution, the lower bound $Q_f^L(\lambda)$ will be given by

$$Q_f^L(\lambda) = 1 - \Phi\left(\frac{\lambda - \mu_{p,0} - \sigma_{p,0}\sqrt{\gamma_{p,\mu}}}{\sigma_{p,0}\sqrt{1 + \gamma_{p,\sigma}}\Phi^{-1}(1 - \alpha)}\right), \quad (6)$$

and it is achieved when $\mu_p = \mu_{p,0} - \sigma_{p,0}\sqrt{\gamma_{p,\mu}}$ and $\sigma_p = \sigma_{p,0}\sqrt{1 + \gamma_{p,\sigma}}$ where $\mu_{p,0}$ and $\sigma_{p,0}^2$ denote the sample estimates of distribution mean $\mu_p$ and variance $\sigma_p^2$ of the received primary signal, respectively. As for uncertainty sets (2) and (3), we have the following optimization problem:

$$\min_{f_1(\xi)} \mathbb{E}_{f_1}[h(\xi, \lambda)] \quad (7a)$$

subject to

$$f_1(\xi) \in U_g(\mu_{p,0}, \sigma_{p,0}, \gamma_{p,\mu}, \gamma_{p,\sigma}). \quad (7b)$$

Since (2) is just a special case of set (3), we only consider set $U_g(\mu_{p,0}, \sigma_{p,0}, \gamma_{p,\mu}, \gamma_{p,\sigma})$ in problem (7a)-(7b). It intends to maximize the cumulative density lies within set $[\lambda, \infty)$, and the decision variable $f_1(\xi)$ is actually the density allocation on each point $\xi$ in its feasible set subject to a series of linear inequality constraints defined in (3). Therefore, problem (7a)-(7b) can be viewed as a linear program with infinite number of decision variables (i.e., densities on each point $\xi \in S$) and finite number of constraints. A straightforward way to solve this problem is to approximate the feasible set $S$ by a finite set of sample points. Then we only consider allocating probability
mass at each of these discrete points. Since simplex method for linear program is very efficient in practice, we can draw a large sample such that the granularity is small enough to ensure desired accuracy. With extreme large samples, this results from linear approximation will be trustworthy and considered as a benchmark for other solution methods.

**B. Equivalent Semi-definite Program**

Besides the finite linear approximation, we try to solve the above problem by an accurate as well as efficient method. Note that problem (7a)-(7b) is a semi-infinite program as the functional variable $f_1(\xi)$ is actually a vector with infinite length. The authors in [16] proposed a method that can transform such semi-infinite program into a convex semi-definite program, which help us determine the detection performance by existing optimization tools.

For the lower detection bound, the worst-case distribution is expected to have a larger variance, thus we consider $E[(\xi - \mu_{p,0})^2] \leq (1 + \gamma_{p,0})^2 p_{opt}$ in constraint (7b). By the Lagrangian method, we have its Lagrangian function in (8) where $\nu, \sigma, \eta$ and $\nu$ are the Lagrangian multipliers associated with different moment constraints. Details are given in the Appendix. Since we pose no constraints on the shape of distribution function $f_1(\xi)$, it is possible to allocate very large mass on a single point. Thus we have the following condition for any $\xi \in S$ when solving the primal problem

$$\nu \xi^2 - 2(\nu + \mu_{p,0}) \nu \xi + \eta \geq -h(\xi, \nu, \eta).$$

(9)

The LHS of (9) is a polynomial of degree two, which defines a quadratic curve $c(\nu, \omega, \eta) = \nu \xi^2 - 2(\omega + \mu_{p,0}) \nu \xi + \eta$ and it should dominate a step function $-h(\xi, \nu, \eta)$, i.e., i.e., $c(\nu, \omega, \eta) \geq -h(\xi, \nu, \eta)$ for any $\xi \in S$. Fig. 1 plots the relation between the quadratic curve and the step function. We may have different curves (e.g., curves $c_1$ and $c_2$) when the coefficients $\nu, \omega, \eta$ take different values. Therefore, condition (9) actually looks for a set of coefficients $\nu, \omega, \eta$ for the quadratic curve such that it always dominates $-h(\xi, \nu, \eta)$. For comparison, we also plot the curve (i.e., $c_3$) for the upper bound of detection probability, which requires $c(\nu, \omega, \eta)$ to dominate $h(\xi, \nu, \eta)$. From this geometric interpretation, we pay attention to two points, i.e., the quadratic curve should be greater than 0 at $\xi = \lambda$ and greater than 1 at $\xi = \mu_{p,0} + \frac{\sigma}{\nu}$.

Then, we can transform problem (7a)-(7b) into a simpler form.

**Proposition 1:** Finding the lower detection bound subject to a distribution uncertainty $U_g(\mu_{p,0}, \sigma_{p,0}, \gamma_{p,0}, \nu_{p,0}, \sigma_{p,0})$ is equivalent to solving a semi-definite program as follows:

$$\begin{align*}
\max_{\eta, \nu, \omega, \eta} & \quad \left[ \mu_{p,0}^2 - (1 + \gamma_{p,0}) \sigma_{p,0}^2 \nu + 2 \mu_{p,0} \omega - \eta - 2 \sigma_{p,0} |\omega| \sqrt{1 + \gamma_{p,0}} \right] \quad \text{(10a)} \\
\text{s.t.} & \quad \nu \lambda^2 - 2(\nu \mu_{p,0} + \omega) \lambda + \eta \geq 0 \quad \text{(10b)} \\
& \quad (\mu_{p,0} - \lambda) \nu + \omega \geq 0 \quad \text{(10c)} \\
& \quad \left[ \eta + 1, \mu_{p,0}, \omega \right] \geq 0. \quad \text{(10d)}
\end{align*}$$

The proof for Proposition 1 is given in Appendix. The first two constraints (10b) and (10c) are linear with respect to variables $\eta$, $\nu$, and $\omega$, while the last constraint (10d) defines the linear matrix inequality. For such a semi-definite program, we have the interior-point algorithm [17] and existing optimization toolbox such as SeDuMi [18] to solve it efficiently.

**C. Bound Achieving Distribution**

An optimal distribution $f_\nu(\xi) \in U_g$ that achieves the detection bound in problem (10a)-(10d) is defined as an extremal distribution. We find that the extremal distribution for problem (10a)-(10d) only assigns positive mass on two points where the quadratic curve $c(\nu, \omega, \eta)$ is tangent with $-h(\xi, \nu, \eta)$. Specifically, the tangential points locate at $\{\lambda, \mu_{p,0} + \frac{\sigma}{\nu}\}$ as indicated by two hollow points in Fig. 1, thus we can construct an extremal distribution $f_{\nu}(\xi)$ that achieves the lower bound as follows:

$$f_{\nu}(\xi) = \begin{cases} 
1 - Q_{\lambda}^L(\lambda), & \text{if } \xi = \lambda \\
Q_{\lambda}^L(\lambda), & \text{if } \xi = \mu_{p,0} + \frac{\sigma}{\nu} \\
0, & \text{otherwise,}
\end{cases}$$

where $Q_{\lambda}^L(\lambda)$ denotes the lower bound of detection probability and is given by the optimum of problem (10a)-(10d). Note that the extremal distribution takes a discrete form. To study its properties, we have the following proposition, which moreover gives a closed-form expression for the lower detection bound.

**Proposition 2:** The extremal distribution achieving the lower bound is with mean and variance as follows:

$$\begin{align*}
\mu_c &= \mu_{p,0} - \sigma_{p,0} \sqrt{\gamma_{p,0}} \\
\sigma_c^2 &= (1 + \gamma_{p,0} - \gamma_{p,0}^2) \sigma_{p,0}^2.
\end{align*}$$

Moreover, given the decision threshold $\lambda \leq \mu_c$, the lower bound can be expressed in an analytical form:

$$Q_{\lambda}^L(\lambda) = \left[ 1 + \sigma_c^2 / (\mu_c - \lambda)^2 \right]^{-1}. \quad \text{(11)}$$

A similar result is presented in [14]. Here we omit the proof for conciseness. The formula in (11) provides an easy way to measure the lower bound, instead of solving the optimization problem in (10a)-(10d).
\[ \Gamma(f_1, \eta, Z, \nu) = \mathbb{E}[h(\xi, \lambda) + \eta + \nu(\xi - \mu_{p,0}^2) - \eta - (1 + \gamma_{p,0})\sigma_{p,0}^2 \nu - \min_{Z} \mathbb{E} \left[ \frac{\kappa_2}{\omega} \mathbb{E} \left[ \frac{\sigma_{p,0}^2}{\xi - \mu_{p,0}^2} - \gamma_{p,0} \sigma_{p,0}^2 \nu - \eta - \gamma_{p,0} s - \sigma_{p,0}^2 \kappa \right] \right] \]

\[ = \max_{Z} \mathbb{E}[h(\xi, \lambda) + \eta - 2(\omega + \mu_{p,0}^2)\xi + \nu\xi^2 + 2\mu_{p,0} \omega + [\mu_{p,0}^2 - (1 + \gamma_{p,0})\sigma_{p,0}^2 \nu - \eta - \gamma_{p,0} s - \sigma_{p,0}^2 \kappa] \]

### IV. A New Uncertainty Model

Up to this point, We present an equivalent problem and a closed-form formula to study the lower detection bound, considering moment uncertainties. Moreover, we show that the lower detection bound is achievable by a discrete distribution. Since the signal samples from channel sensing hardly follow a discrete distribution, the resulting detection bound may be rather conservative in practice. In this part, we intend to obtain less conservative bounds by extracting a reference distribution, rather than just signal statistics, from historical data. Though signal distribution is fluctuating over time and hard to describe in a closed-form expression, we may consider empirical distribution as a useful reference and allow the actual signal distribution to shift around it. For example, we can assume that the noise signal \( f_0(\xi) \) is more or less close to a known Gaussian distribution \( g_0(\xi) \), which can be obtained based on long-term field measurement. Similarly, primary signal’s reference distribution \( g_p(\xi) \) can take different closed-form expressions as in [5], [8] when channel conditions fluctuate.

#### A. Probability Distance Measure

The difference between \( f_0(\xi) \) (or \( f_1(\xi) \), respectively) and its reference \( g_n(\xi) \) (or \( g_p(\xi) \), respectively) can be described by a probabilistic distance measure. For example, the Kullback-Leibler (KL) divergence [19] is a non-symmetric measure of the difference between two probability distributions, i.e., \( f(\xi) \) and \( g(\xi) \). Generally, one of the distribution \( f(\xi) \) represents the signal’s real distribution through precise modeling. The reference \( g(\xi) \) is a closed-form approximation based on theoretic assumptions and simplifications. The definition of KL divergence between two continuous distributions is given as follows:

\[ D_{KL}(f(\xi), g(\xi)) = \int_{\xi \in S} [\ln f(\xi) - \ln g(\xi)] f(\xi) d\xi. \quad (12) \]

When distributions \( f(\xi) \) and \( g(\xi) \) are close to each other, the distance measure is close to zero. Based on the KL divergence, we thus define the distribution uncertainty as follows:

\[ U_r(g(\xi), D_0) = \{f(\xi) \mid \mathbb{E}f[\ln f(\xi) - \ln g(\xi)] \leq D_0\}, \quad (13) \]

where \( D_0 > 0 \) represents a distance limit and is obtained from empirical data or real-time measurement. It indicates signal’s fluctuation level. If the signal is highly volatile, we have less confidence on the reference distribution and thus set a larger distance limit. In practice, we can set \( g_n(\xi) \) and \( g_p(\xi) \) as those distributions commonly employed in literature [5]–[8] and maintain the distance limit during channel measurements. Each time we fit the sensed signal samples into a closed-form distribution and calculate the KL divergence with respect to \( g_n(\xi) \) (or \( g_p(\xi) \), respectively) if PU is absent (or present, respectively), then we overwrite the stored distance limit by the new KL divergence if it becomes larger. The reference distributions can also be updated online if real-time measurements indicate large deviations, in terms of KL divergence, from their reference distributions.

#### B. False Alarm Probability with Fixed Decision Threshold

Consider the noise signal \( f_0(\xi) \) with reference distribution \( g_n(\xi) \) and distance limit \( D_n \), we have the following constraints for noise distribution \( f_0(\xi) \):

\[ \mathbb{E}[\ln f_0(\xi) - \ln g_n(\xi)] \leq D_n \quad (14a) \]
\[ \mathbb{E}[f_0(1)] = 1. \quad (14b) \]

By the Lagrangian method, we have the worst-case false alarm probability \( Q_f^G(\lambda) \) as follows:

\[ Q_f^G(\lambda) = \min_{\tau, \eta} \mathbb{E}[h(\xi, \lambda) - \tau \ln f_0(\xi) g_n(\xi)] + \tau D_n + \eta \]

where \( \tau \geq 0 \) and \( \eta \) are Lagrangian multipliers associated with constraints (14a) and (14b), respectively. Let \( P(\lambda, f_0, \tau, \eta) = \mathbb{E}[h(\xi, \lambda) - \tau \ln f_0(\xi) g_n(\xi)] \), then the derivative of \( P(\lambda, f_0, \tau, \eta) \) with respect to \( f_0(\xi) \) is given by

\[ \frac{\partial P}{\partial f_0} = \int_{\xi \in S} \left( h(\xi, \lambda) - \tau \ln f_0(\xi) g_n(\xi) - \tau - \eta \right) d\xi. \]

By the Karush-Kuhn-Tucker (KKT) optimality conditions, we thus have:

\[ h(\xi, \lambda) - \tau \ln f_0(\xi) g_n(\xi) - \tau - \eta = 0 \quad (15a) \]
\[ \int_{\xi \in S} f_0(\xi) d\xi = 1 \quad (15b) \]
\[ D_n - \mathbb{E}\left[ \ln f_0(\xi) g_n(\xi) \right] \geq 0 \quad (15c) \]
\[ \tau \left( D_n - \mathbb{E}\left[ \ln f_0(\xi) g_n(\xi) \right] \right) = 0. \quad (15d) \]

From (15a), the optimal distribution function is as follows:

\[ f_0^*(\xi) = g_n(\xi) \exp \left( \frac{h(\xi, \lambda) - \tau - \eta}{\tau} - 1 \right). \quad (16) \]

The dual variables \( (\tau, \eta) \) in (16) should be chosen properly such that conditions (15b)-(15d) are satisfied. Specifically, we have the following results:

### Proposition 3:

The choice of \( (\tau, \eta) \) is a solution of the following nonlinear equations:

\[ H_1(\tau, \eta) = R(\lambda) e^{-\eta / \tau} + S(\lambda) (e^{1 - \eta / \tau} - 1) = 0 \quad (17a) \]
\[ H_2(\tau, \eta) = S(\lambda) e^{(1 - \eta / \tau)} - \eta - \tau (1 + D_n) = 0. \quad (17b) \]

where \( R(\lambda) = G_n(\lambda) e^{-1} \), \( S(\lambda) = (1 - G_n(\lambda)) e^{-1} \), and \( G_n(\lambda) = \int_{\xi \leq \lambda} g_n(\xi) d\xi \) denotes the cumulative distribution function of reference distribution \( g_n(\xi) \).
The proof for Proposition 3 is straightforward by substituting the optimal distribution \( f^*_p(\xi) \) back to (15b)-(15d). Details are omitted here due to space limit. However, it is still very hard to obtain an explicit solution from (17a) and (17b), thus we propose the Newton iterations as detailed in Algorithm 1.

### C. Determine a Robust Decision Threshold

Once we determine the solution for (17a) and (17b) in Proposition 3, we obtain the worst-case false alarm probability from (15a) and (15d) as follows:

\[
Q^U_j(\lambda) = \mathbb{E}_{f^*_p}[h(\xi, \lambda)] = (1 + D_n)\tau + \eta. \tag{18}
\]

Then our task is to find the decision threshold \( \lambda \) such that \( Q^U_j(\lambda) = \alpha \), which involves the calculation of inverse function of \( Q^U_j(\lambda) \) and it is not directly possible from (18). However, we have the following property regarding function \( Q^U_j(\lambda) \) that may help us design a search method.

**Proposition 4:** Worst-case false alarm probability \( Q^U_j(\lambda) \) is non-increasing with respect to the decision threshold \( \lambda \).

**Algorithm 1** Search for robust decision threshold

**Input:** Reference distribution \( g_0(\xi) \), distance limit \( D_n \), search radius \( \rho \), and tolerance \( \epsilon \)

**Output:** Robust decision threshold such that \( Q^U_j(\lambda^*) = \alpha \)

1. set \( \lambda_0 = 0 \), \( \lambda^* = \rho \mu_{p,0} \), and \( H(\tau, \eta) = [H_1, H_2]^T \)
2. while \( |\lambda_- - \lambda^*| > \epsilon \)
3. set \( \lambda = \frac{\lambda_- + \lambda^*}{2} \) and initiate the time iteration \( k = 1 \)
4. while \( |H(\tau_k, \eta_k)| > \epsilon \)
5. evaluate \( H(\tau_k, \eta_k) \) and Jacobian matrix \( J(\tau_k, \eta_k) \)
6. solve \( J(\tau_k, \eta_k)\Delta x_0 = -H(\tau_k, \eta_k) \)
7. update \( \tau_{k+1} = [\tau_k + \Delta \tau_k]^+, \eta_{k+1} = \eta_k + \Delta \eta_k \)
8. update \( Q^U_j(\lambda) = (1 + D_n)\tau_{k+1} + \eta_{k+1} \)
9. set \( k = k + 1 \)
10. end while
11. if \( (Q^U_j(\lambda) - \alpha)(Q^U_j(\lambda_-) - \alpha) < 0 \)
12. then set \( \lambda^- = \lambda \) else set \( \lambda_+ = \lambda \) end if
13. if \( |Q^U_j(\lambda) - \alpha| < \epsilon \) break end if
14. end while
15. set \( \lambda^* = \lambda \)

The conclusion in Proposition 4 is obvious since we have \( dQ^U_j(\lambda)/d\lambda = d\mathbb{E}_{f^*_p}[h(\xi, \lambda)]/d\lambda = -f^*_p(\lambda) \leq 0 \). Though direct solution for \( Q^U_j(\lambda) \) is \( \alpha \) is not available, the monotonicity of \( Q^U_j(\lambda) \) enlightens us a bisection method to search the solution for \( Q^U_j(\lambda) = \alpha \). We propose the search procedure in Algorithm 1. It performs the search within an interval \( [0, \rho \mu_{p,0}] \), where \( \rho \) is an empirical constant such that \( Q^U_j(\rho \mu_{p,0}) < \alpha \). From lines 2 to 10 of Algorithm 1, we use the Newton iterations to solve the equations in Proposition 3 and obtain the worst-case false alarm with fixed decision threshold\(^2\). Then we compare the false alarm probabilities at \( \lambda \) and \( \lambda_- \) with the required false alarm probability \( \alpha \), respectively. The comparing result helps to shrink the search region as in lines 11-13.

\(^2\)In the update of dual variables, we use \( \lfloor x \rfloor = \max\{x, 0\} \) to denote the projection on non-negative real number.

### D. Lower Bound of Detection Probability

Given the decision threshold in Algorithm 1, we proceed to analyze the detection bound with distance limit \( D_p \) and reference distribution \( g_p(\xi) \), which can be a Rayleigh or Rician distribution depending on different channel models. We have assumed that the real distribution \( f(\xi) \) is not very different from the reference distribution, however, when we actually detect a very large deviation from the reference distribution, we can update \( g_p(\xi) \) as the most suitable one from a set of candidate models by goodness of fit test. With this assumption, we consider the reference model as the intersection of moment uncertainty and a reference distribution, i.e., \( \mathcal{U}_r(g_p(\xi), D_p) \cap \mathcal{U}_g(\mu_{p,0}, \sigma_{p,0}, \gamma_{p,\mu}, \gamma_{p,\sigma}) \). Then, we can obtain the KKT conditions for the lower bound problem by the same way as in (15a)-(15d):

\[
\tilde{h}(\xi, \lambda) + \tau \ln \frac{f^*_p(\xi)}{g_p(\xi)} + \tau = 0 \tag{19a}
\]

\[
\tau \left( \mathbb{E}_p \left[ \ln \frac{f^*_p(\xi)}{g_p(\xi)} - D_p \right] \right) = 0 \tag{19b}
\]

\[
\nu \left( \mathbb{E}_p \left[ \xi^2 - 2\mu_{p,0} \xi \right] + \mu_{p,0}^2 - (1 + \gamma_{p,\sigma})\sigma_{p,0}^2 \right) = 0 \tag{19c}
\]

\[
\omega \left( \mathbb{E}_p \left[ -2\xi + 2\mu_{p,0} - 2\sigma_{p,0} \text{Sign}(\omega) \sqrt{\gamma_{p,\mu}} \right] \right) = 0 \tag{19d}
\]

where \( \tilde{h}(\xi, \lambda) \triangleq h(\xi, \lambda) - 2(\omega + \mu_{p,0}) \xi + \nu \xi^2 \) and \( (\eta, \nu, \omega, \tau) \) are the Lagrangian multipliers. Then we obtain the optimal distribution function as follows:

\[
f^*_p(\xi) = g_p(\xi) \exp \left( \frac{1}{\tau} \left( h(\xi, \lambda) \right) - 1 \right). \tag{20}
\]

If \((\eta, \nu, \omega, \tau)\) satisfy the equations (19a)-(19e), we can obtain the lower bound of detection probability \( Q^U_j(\lambda) = \mathbb{E}_{f^*_p}[h(\xi, \lambda)] \) as follows:

\[
Q^U_j(\lambda) = \mathbb{E}_p \left[ \xi^2 - 2\mu_{p,0} \xi \right] + \mu_{p,0}^2 - (1 + \gamma_{p,\sigma})\sigma_{p,0}^2 \}
\]

\[
\omega \left( \mathbb{E}_p \left[ -2\xi + 2\mu_{p,0} - 2\sigma_{p,0} \text{Sign}(\omega) \sqrt{\gamma_{p,\mu}} \right] \right) = 0 \tag{21}
\]

However, it is very complicated to transform the KKT conditions (19a)-(19e) into a compact form similar to that in Proposition 3, and we are unable to solve \((\eta, \nu, \omega, \tau)\) by Newton iterations as in Algorithm 1. Here, we try to solve it in another way: At some iteration \( k \), we fix \((\eta_k, \nu_k, \omega_k, \tau_k)\) and find the optimal distribution function in (20). Then we determine the numerical approximations for signal moments \( \mathbb{E}[1], \mathbb{E}[\xi], \mathbb{E}[\xi^2] \), and \( \mathbb{E} \left[ \ln \frac{f^*_p(\xi)}{g_p(\xi)} \right] \), respectively. These values are used to check whether (19b)-(19e) are satisfied. If not satisfied, we will update \( \xi_k = (\eta_k, \nu_k, \omega_k, \tau_k) \) in a direction \( \Delta_k = [\Delta_\eta, \Delta_\nu, \Delta_\omega, \Delta_\tau] \) given as follows:

\[
\Delta_\eta = \int_{\xi \in S} f^*_p(\xi) d\xi - 1 \tag{22a}
\]

\[
\Delta_\nu = \mathbb{E}_p[\xi^2 - 2\mu_{p,0}\xi + \mu_{p,0}^2 - (1 + \gamma_{p,\sigma})\sigma_{p,0}^2] \tag{22b}
\]

\[
\Delta_\omega = -2\mathbb{E}_p[\xi - 2\mu_{p,0} - 2\mu_{p,0}\text{Sign}(\omega)\sqrt{\gamma_{p,\mu}}] \tag{22c}
\]

\[
\Delta_\tau = \left[ \mathbb{E}_p \left[ \ln \frac{f^*_p(\xi)}{g_p(\xi)} - D_p \right] \right]^+ \tag{22d}
\]
Then we have \( x_{k+1} = x_k + \text{diag}(\beta(k)) \Delta_k \), where vector \( \beta(k) \) denotes a sufficiently small step-size at \( k \)-th iteration for each term in \( x_k \). With this new variable \( x_{k+1} = (\eta_{k+1}, \nu_{k+1}, \omega_{k+1}, \tau_{k+1}) \), we return back to update the distribution function as in (20). This process iterates until \( x_k \) converges to some \( x_\infty \), and finally returns the lower detection bound by substituting \( x_\infty \) into (21). As a summary, we compare all different models in Table I, where RUM denotes the reference uncertainty model and LDB means lower detection bound.

**TABLE I: Comparison between uncertainty models**

<table>
<thead>
<tr>
<th>Model</th>
<th>Dist. type</th>
<th>Dist. parameter</th>
<th>LDB</th>
<th>Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>KDUS</td>
<td>known</td>
<td>unknown</td>
<td>(6)</td>
<td>Model sensitive</td>
</tr>
<tr>
<td>GDES</td>
<td>unknown</td>
<td>known</td>
<td>(11)</td>
<td>Conservative</td>
</tr>
<tr>
<td>GDUS</td>
<td>unknown</td>
<td>unknown</td>
<td>(11)</td>
<td>Very conservative</td>
</tr>
<tr>
<td>RUM</td>
<td>unknown</td>
<td>unknown</td>
<td>(21)</td>
<td>Less conservative</td>
</tr>
</tbody>
</table>

V. Numerical Results

In the simulations, we first obtain the robust decision thresholds under different uncertainties, and verify their robustness by simulated noise signals. Specifically, we emulate noise fluctuations using random variables that are drawn from different distributions. Then, we count the number of events when signal sample is greater than the decision threshold, and use it to estimate the actual false alarm probability. We set \( \alpha = 0.1 \) as the required false alarm probability, and define \( \gamma_{n,\mu} = \gamma_{n,\sigma} = 0.1 \) to regulate the size of noise uncertainty. The validation of detection bounds contains two parts. For the moment uncertainties, we set \( \mu_{p,0} = 3 \) and \( \sigma_{p,0}^2 = 1 \) as the sample mean and variance of the received primary signal, respectively, and \( \gamma_{p,\mu} = \gamma_{p,\sigma} = 0.1 \) to regulate their fluctuating ranges, respectively. Then we compare the lower detection bounds obtained through finite linear approximations, equivalent semi-definite program, and closed-form formula in (11), respectively. As a benchmark method, the finite linear approximation will help us identify the correctness of the semi-definite equivalence and formula in (11). For the reference uncertainty model, we assume a known reference distribution, which can be channel-dependent and updated online when spectrum environment changes. Without lose of generality, we assume that the reference distributions of noise and primary signals both follow Gaussian distributions, then we numerically study the false alarm probability and detection probability by our proposed iterative procedures. Their convergent values are also compared with that of the moment uncertainties.

A. False Alarm Probability under Noise Uncertainties

When we consider the KDUS model in (1), we obtain a larger detection threshold in (5) which we denote as \( \lambda_{KDUS} \). Compared with a nominal design\(^3\) \( \lambda_{DSM} = \mu_{n,0} + \sigma_{n,0}^2 \Phi^{-1}(1-\alpha) \). Fig. 2 plots the maps from decision thresholds to false alarm probabilities under deterministic signal model (DSM) and the KDUS model, respectively. Though we did not analyze the false alarm probability with GDES and GDUS models, the formulation in (7a)-(7b) can be slightly modified to study the worst-case false alarm probability. We also plot their results in Fig. 2 for comparison. Note that \( \lambda_{DSM} < \lambda_{KDUS} < \lambda_{GDES} < \lambda_{GDUS} \), which implies that a larger decision threshold is required to ensure the same false alarm probability if signal’s distribution information is unavailable.

To investigate the actual false alarm probability with these decision thresholds, we emulate noise signal \( f_0(\xi) \) by a stream of random samples that are generated from different Gaussian distributions. The noise signal’s mean and variance are confined by parameters \( \gamma_{n,\mu} \) and \( \gamma_{n,\sigma} \), respectively. Therefore, the actual noise samples are randomly drawn from any Gaussian distribution \( N(\mu_n, \sigma_n^2) \) with \( \mu_n = \mu_{n,0} - \sqrt{\mu_{p,0}^2 + \sigma_{p,0}^2 - \gamma_{p,\mu} \sigma_{n,0}} \leq \mu_n \leq \mu_{n,0} + \sqrt{\gamma_{p,\mu} \sigma_{n,0}} \) and \( \sigma_n^2 = (1 - \gamma_{n,\sigma}) \sigma_{n,0}^2 \leq \sigma_n^2 \leq \sigma_{n,0}^2 (1 + \gamma_{n,\sigma}) \). Then we apply the emulated noise signal to different decision thresholds, and check the actual false alarm probabilities. Given different required false alarm probabilities (\( x \)-axis in Fig. 3), we plot the actual false alarm probabilities with different decision thresholds in Fig. 3. In case I, the noise mean \( \mu_n \) and variance \( \sigma_n^2 \) are uniformly drawn from their fluctuating ranges, respectively. We can see that, the required false alarm probability can be guaranteed by decision threshold \( \lambda_{DSM} \) on average, however, it is violated in case II where the noise signal tends to take the largest mean \( \mu_{n,0} + \sqrt{\mu_{p,0}^2 + \sigma_{p,0}^2} \) and variance \( \sigma_{n,0}^2 (1 + \gamma_{n,\sigma}) \). Since the KDUS model considers worst-case signal fluctuation in designing the decision threshold, it still guarantees the actual false alarm probability to be less than the required level in case II.

B. Detection Bounds with Moment Uncertainties

To study the detection probability, we fix the decision threshold as \( \lambda_{KDUS} \). In Fig. 4, we first plot the lower detection bound with KDUS model, which is explicitly given in (6). Regarding the GDUS model in problem (7a)-(7b), we have discussed two solution methods. The first method considers a finite linear approximation of the original semi-infinite linear program by dividing the feasible set into discrete points with a grid size \( \Delta \). While the second method finds the exact lower

\(^3\) It means that we assume a deterministic signal model, and denote the resulting decision threshold as \( \lambda_{DSM} \).
C. The Importance of a Reference Distribution

We assume that the reference distribution of noise signal is a standard Gaussian distribution \( g_n(\xi) \). Considering the same parameters \( \gamma_{n,\mu} = \gamma_{n,\sigma} = 0.1 \) as in Section V-A, we find that the KL divergence between \( g_n(\xi) \) and any distribution \( f_0(\xi) \) in \( U_d(0,1,0.1,0.1) \) is always less than 0.076. Therefore, we set the distance limit \( D_n \) in an equivalent magnitude, e.g., \( D_n = 0.1 \). For different choices of decision threshold \( \lambda \), we solve a pair of variables \( (\tau, \eta) \) that satisfies equations (17a) and (17b). Then we obtain the worst-case false alarm probability \( Q_f^U(\lambda) = (1 + D_n)(\tau + \eta) \) for three different distance limits \( D_n \), which are plotted in Fig. 5, respectively. We also plot false alarm probabilities with different moment uncertainties in the same figure. It is obvious that decision threshold with the reference model increases with the increase of the distance limit. This observation is intuitive since larger distance limit allows the noise signal to fluctuate more intensively. We also note that the false alarm probability is a decreasing function of the decision threshold. Thus, we are justified to use the bisection method in Algorithm 1 to search the decision threshold that satisfies the false alarm probability requirement. Comparing with moment uncertainties, the reference model produces larger decision thresholds than that of the KDUS model and less than that of the GDES and GDUS models.

A comparison in Fig. 6 also shows a similar trend that the detection bound of a reference model is looser than that of the KDUS model, while tighter than that of the GDES and GDUS models. This is because, the KDUS model requires the signal to follow exactly a known distribution, while the reference model considers a more general case. It allows discrepancy between actual distribution and its reference, however, the discrepancy is limited and confined by a probabilistic distance measure, which poses a stronger assumption on the shape of distribution functions than that of the GDES and GDUS models.

Simply put, the reference model allows the actual signal to follow a different distribution function, which should not be too abnormal based on historical data or empirical knowledge. In the simulation, we emulate primary signal by two streams of random samples. One is drawn from a Gaussian distribution (denoted as Gaussian in Fig. 6) whose mean and variance are uniformly picked from sets \([\mu_{n,0} - \sqrt{\gamma_{n,\mu}}\sigma_{n,0}, \mu_{n,0} + \sqrt{\gamma_{n,\mu}}\sigma_{n,0}]\) and \([1 - \gamma_{p,\sigma}][\sigma_{p,0}^2, (1 + \gamma_{p,\sigma})\sigma_{p,0}^2]\), respectively. The other one follows a log-normal distribution (denoted as Lognormal in Fig. 6) with mean and variance bounded in the same uncertainty set. Since the KDUS model assumes Gaussian distribution when calculating the detection bound, it well bounds the Gaussian signal as in Fig. 6. However, it
fails to bound it when the actual signal follows a log-normal distribution. Fortunately, our reference model provides tighter bound in this case since it is tolerable to the mismatch of distribution functions.

VI. CONCLUSION
In this paper, we study the sensing performance with fluctuating signals, which are characterized by different uncertainty models. First, we find a robust decision threshold that guarantees the required false alarm probability under noise fluctuation. Then we consider the distribution uncertainty of PUs’ signals and study the sensing performance in terms of the lower detection bound. For the moment uncertainties, we show that the lower detection bound is equivalent to a semi-definite program, and present a closed-form formula to calculate the detection bound. Furthermore, we introduce a reference distribution into the signal uncertainties. This new model allows signal distribution to be different at each observation, however, not to be very abnormal with respect to the past observations or empirical knowledge. Numerical results show that the reference model is flexible to characterize signal’s fluctuation, and could improve the detection performance compared with moment uncertainties in general case.

APPENDIX
For ease of analysis, we first rewrite the second constraint in (3) as
\[
\mathbb{E} \left[ \begin{array}{c}
\sigma_{p,0}^2 \\
\xi - \mu_{p,0}
\end{array} \right] \geq 0
\]
then we assign different dual variables \( \eta, Z \geq 0 \), and \( \nu > 0 \) to these constraints, respectively. Let \( Z = \begin{bmatrix} \kappa & \omega \\ \omega & s \end{bmatrix} \). We have the Lagrangian function \( \Gamma(f_1, \eta, Z, \nu) \) as in (8). By the geometric interpretation in Fig. 1, we then have the dual problem:

\[
\max_{Z, \nu, \tau} \left[ \frac{1}{2} \sigma_{p,0}^2 - (1 + \gamma_{p, \sigma}) \sigma_{p,0}^2 \right] \nu \\
+ 2\mu_{p,0} \omega - \eta - \gamma_{p, \mu}s - \sigma_{p,0}^2 \kappa \\
(\mu_{p,0} \omega + \omega^2 - \nu (\eta + 1)) \leq 0, \\
\nu \eta^2 - 2(\omega + \mu_{p,0} \nu) \xi + \eta \geq 0, \\
\forall \eta \leq \lambda, \\
Z \geq 0, \ \nu \geq 0,
\]

where \( Z \geq 0 \) implies \( \kappa s - \omega^2 \geq 0, \kappa \geq 0 \) and \( s \geq 0 \).

If \( s = 0 \), we have \( \kappa = \omega = 0 \), then (23a) is reduced to
\[
\left[ \frac{1}{2} \sigma_{p,0}^2 - (1 + \gamma_{p, \sigma}) \sigma_{p,0}^2 \right] \nu - \eta = 0. If s > 0, then we have
\[
\left[ \frac{1}{2} \sigma_{p,0}^2 - (1 + \gamma_{p, \sigma}) \sigma_{p,0}^2 \right] \nu + 2\mu_{p,0} \omega - \eta - \gamma_{p, \mu}s - \sigma_{p,0}^2 \kappa \\
\leq 2\mu_{p,0} \omega - \eta - 2\sqrt{\gamma_{p, \mu}} \sigma \omega.
\]

Let \( Q^2_d (\lambda) = \left[ \frac{1}{2} \sigma_{p,0}^2 - (1 + \gamma_{p, \sigma}) \sigma_{p,0}^2 \right] \nu + 2\mu_{p,0} \omega - \eta - 2\sqrt{\gamma_{p, \mu}} \sigma \omega. \) If \( \mu_{p,0} + \frac{s}{\omega} \leq \lambda, \) then (23c) is reduced to
\[
(\mu_{p,0} \omega + \omega^2 - \nu \eta)^2 - 2\omega \sqrt{\gamma_{p, \mu} \sigma} - \frac{s}{\omega} < 0.
\]

where a negative probability in this case is not feasible in practice, we consider the alternate case which leads to the equivalence in Proposition 1.

REFERENCES

Fig. 6: Lower detection bounds with different uncertainties.