Ordering of Self-Organizing Maps in Multidimensional Cases

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It has been proved that in one-dimensional cases, the weights of Kohonen’s self-organizing maps (SOM) will become ordered with probability 1; once the weights are ordered, they cannot become disordered in future training. It is difficult to analyze Kohonen’s SOMs in multidimensional cases; however, it has been conjectured that similar results seem to be obtainable in multidimensional cases. In this note, we show that in multidimensional cases, even though the weights are ordered at some time, it is possible that they become disordered in the future.

1 Introduction

Biologically motivated self-organizing maps (SOM) have been the focus of much interesting neural network research. In what follows we focus on Kohonen’s SOMs (Kohonen, 1989, 1995).

Consider an SOM with n output neurons where the ith neuron has initial weight $w_i = \{w_{i1}, \ldots, w_{im}\}^T$. The feature map formation of the SOM follows an iterative procedure. At time $t$, a pattern $x = \{x_1, \ldots, x_m\}^T$ enters the network. Neuron $c$ (the winner), whose weight $w_c$ is metrically nearest to the pattern $x$, is selected,

$$\|w_c - x\| = \min_{s \in \{1, \ldots, n\}} \|w_s - x\|, \quad (1.1)$$

and the weights of all neurons are then changed according to the updating rule,

$$w_s(t + 1) = \begin{cases} w_s(t) + \eta \Lambda(c, s)(x - w_s) & \text{for } s \in N_c(t) \\ w_s(t) & \text{others} \end{cases}, \quad (1.2)$$

where $\eta = \eta(t)$ is the learning rate ($0 \leq \eta \leq 1$). The function $\Lambda(c, s)$ is
called the neighborhood function, which decreases with increasing distance between \( s \) and \( c \).

Two important problems arise concerning the ordering of weights during the learning process of an SOM: (1) Can the weights of an SOM be ordered through self-organizing? (2) Once the weights of an SOM are ordered, do they always remain ordered in future training time? For the one-dimensional case, the answers to the two problems are positive (Kohonen, 1995; Cottrel & Fort, 1987; Erwin, Obermayer, & Schulten, 1992). However, extending these results to the multidimensional case has been difficult at best. Kohonen (1995) has conjectured that in multidimensional cases, similar results seem to be obtainable. Budinich (1995) explains intuitively why these proofs on convergence to ordered states in one-dimensional case do not extend to multidimensional cases. In this note, we further prove that in the multidimensional case, even if the weights of an SOM are ordered at some time, it cannot be guaranteed that they will always remain ordered.

2 Ordering in Multidimensional Cases

In multidimensional cases, there exist no simple relevant position relations among vector points, and how to define neighborhood relationships is still a problem for SOMs. In order to be visualized, the neighboring weights can be connected (mostly in rectangular or hexagonal array) using straight line segments (Kohonen, 1989; Ritter & Schulten, 1986; Kangas, Kohonen, & Laaksonen, 1990). Normally we can have the following definition.

**Definition 1.** A simplex of a feature map is a closed field formed by least straight line segments connecting neighboring weights of an SOM. The weights of an SOM are called ordered if all the simplexes formed by these weights are nonintersecting.

For the two-dimensional case, ordering of weights appears when no straight line segments connecting neighboring weights intersect each other.

In fact, in order to prove our conclusions on Kohonen’s conjecture, we need only consider the following reasonable case, which can occur in the applications of Kohonen’s SOMs.

**Case 1.** At time \( t \), the weights of an SOM are ordered. A new pattern \( x \) enters the SOM and is within a simplex \( M \), but the weight \( w_{c_1}(t) \), which is nearest to the pattern \( x \), is outside \( M \), and at least one neuron, \( c_{i_0} \), which is outside \( M \), is not in \( N_{c_1}(t) \).

\(^1\) The neighborhood function can be simply represented by \( \Lambda(c, s) \) in this note, but it should be noted that \( \Lambda(c, s) = \Lambda(r_c, r_s) \), where vector \( r_i \) represents the \( i \)th neuron’s coordinate (the \( i \)th neuron’s position) in cortical space.
Suppose at time $t$ there is a feature map as shown in Figure 1a, where the weights have been ordered. At time $t + 1$ when the new pattern $x$ enters the SOM, the weight $w_{c_1}(t)$ and its neighbors $w_{c_5}(t)$, $w_{c_6}(t)$, and so on, will move toward the pattern $x$ according to the learning rule (see equation 1.2). After weight adjustments the simplex $M$ formed by $w_{c_5}(t)$, $w_{c_6}(t)$, ... and $w_{c_{10}}(t)$ will change into a new simplex $M'$ formed by $w_{c_5}(t + 1)$, $w_{c_6}(t + 1)$, ... and $w_{c_{10}}(t + 1)$ (in Figure 1c some of the weights forming $M$ are shown unchanged at this time). Obviously $x$ is also within $M'$, and $w_{c_1}(t)$ is outside $M'$. The line segment between $w_{c_1}(t)$ and $x$ will intersect one side of $M'$. Suppose it intersects the side $w_{c_5}(t + 1)w_{c_{10}}(t + 1)$ and the cross point is $p$.

As stated in Gasson (1983), if the point $p$ divides the line segment $w_{c_1}(t)x$ in the ratio $\lambda : (1 - \lambda)$,

$$\lambda = \frac{w_{c_1}(t)p}{w_{c_1}(t)x},$$

then the point $p$ can be denoted as $p = w_{c_1}(t) + \lambda(x - w_{c_1}(t))$. Given the weights $w_{c_5}(t)$, $w_{c_{10}}(t)$, and $w_{c_1}(t)$ and the pattern $x$ we can get the solution of $\lambda$.

$$\lambda = \text{Row} \left( \begin{bmatrix} (x - w_{c_1}(t)) & (w_{c_{10}}(t + 1) - w_{c_5}(t + 1)) \end{bmatrix}^{-1} \begin{bmatrix} w_{c_{10}}(t + 1) - w_{c_1}(t) \end{bmatrix}, 1 \right)$$

(2.1)

where $\text{Row}(A, i)$ represents the $i$th element of vector $A$.

Since

$$\lambda = \frac{w_{c_1}(t)p}{w_{c_1}(t)x},$$

for any small positive value $\epsilon > 0$, i.e., $\epsilon = \eta(t)\Lambda(0)$, it is possible that $\lambda < \epsilon$ in some applications. (For example, if $x$ is input into the SOM repeatedly the weight of $c_1$ can move to $x$ according to the learning rule (in equation 2) with any small distance between $w_{c_1}(t)$ and $x$.) Because $w_{c_1}(t + 1) = w_{c_1}(t) + \eta(t)\Lambda(0)(x - w_{c_1}(t))$, if $\eta(t)\Lambda(0) > \lambda$, the weight $w_{c_1}(t + 1)$ lies within $M'$. According to Case 1, since $c_{i_0}$, which is outside $M$, is not in $N_{c_1}(t)$, $w_{c_{i_0}}(t + 1) = w_{c_{i_0}}(t)$, and thus, $w_{c_{i_0}}(t + 1)$ is outside $M'$. It is obvious that there exists a sequence of neurons $c_{i_0}, c_{i_1}, c_{i_2}, \ldots, c_k = c_1$ such that $c_{i_j}$ is one of neighboring neurons of $c_{i_{j-1}}$ ($j = 0, \ldots, k - 1$). Because $w_{c_{i_0}}(t + 1)$ is outside $M'$ and $w_{c_{i_k}}(t + 1)$ within it there exists $l$ such that $w_{c_{i_l}}(t + 1)$ is outside $M'$ but $w_{c_{i_{l+1}}}(t + 1)$ lies within $M'$, and thus the straight line segment connecting neighboring weights $w_{c_{i_1}}(t + 1)$ and $w_{c_{i_{l+1}}}(t + 1)$ will intersect one side of $M'$ and the weights will become disordered.

Case 1 can also appear in multidimensional cases with any type of lattice even though the input pattern and the relative winner lies within the same simplex. Thus, we have the following theorem on the ordering of Kohonen’s SOMs.
Theorem 1. In multidimensional cases for any type of lattice and any small learning rate, even though the weights of an SOM have been ordered, it is possible that there exists a sequence of input patterns and their relative occurrence frequencies so that these weights become disordered at some future time.

References


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