Proc. R. Soc. Lond. A (1999) 455, 3351–3361

Effect of fibre shape on the stresses within fibres in fibre-reinforced composite materials

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Short title: Fibre shape in composite materials

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Word count: 2972

Abstract

Equations have been derived for calculating the stress distributions in a tapered reinforcing fibre in a composite material, i.e. for a fibre which does not have a uniform radius. In general, deformation of the matrix in a composite induces a radial (compressive) stress at the fibre surface and an axial (tensile) stress. The equations were solved for a circular conical, paraboloidal and ellipsoidal fibre embedded in a plastic matrix. Results were compared with the familiar results for a uniform cylindrical fibre (i.e. with a constant radius) for which the radial stress at the surface is zero. For a uniform cylinder, the axial stress increases linearly, from zero at the ends, to a maximum value at the centre. At the other extreme, the axial stress in a conical fibre was shown to be constant. The intermediate cases of a paraboloidal and an ellipsoidal fibre showed axial stress distributions lying between these two extremes.

Keywords: fibre shape, fibre composite materials

1. Introduction

Reinforcement of materials by fibres with high tensile stiffness and strength allows the production of composite materials of high strength and toughness (Kelly and Macmillan 1986). The same principles are exploited in animal tissues where collagen fibrils provide reinforcement (Hukins 1982). Recently, it has been noticed that collagen fibrils may not be rod-shaped and they have been described as having "pointed" or "parabolic" ends (Holmes *et al*. 1992; Trotter and Purslow 1992; Trotter *et al*. 1994). These descriptions are not based on a detailed analysis of the fibril shapes but they indicate that the fibrils are not cylindrical.

This paper presents an investigation of the advantages of reinforcing composite materials whose fibres have tapering ends. The specific tapered shapes considered are a circular cone, a paraboloid and an ellipsoid, because they can be described analytically. A general model has been formulated for reinforcement of composite materials. The model has been applied to the case of a plastic matrix deforming under an applied tensile load. Deformation of the matrix generates an interfacial shear stress which applies a stress to the fibre (Kelly and Tyson 1965). The reason for applying our general equation to the special case of a plastic matrix is discussed in section 3(a). In a fibre with tapered ends, this generates a radial compressive stress as well as an axial tensile stress. A comparison with the stress distribution in a uniform cylindrical fibre enables the effect of each fibre shape to be assessed.

2. Formulation of a general model

(a) Description of the fibre composite system

Figure 1 shows our model for a single fibre composite material described using a cylindrical polar coordinate system (r, ϕ, z). In this paper r denotes the radius of the fibre which may vary along its length; ϕ is the azimuthal angle of the coordinate system. The axis and centre of the fibre define the z-axis and the origin, O, of the coordinate system, respectively. The fibre is homogeneous, has a length of $2L$ and is entirely embedded in a matrix. Note that only one half of the length of the fibre is shown because it is symmetrical about the origin.

[Insert figure 1 here]

In a discontinuous fibre-reinforced composite, an external applied stress deforms the matrix material. This exerts an interfacial shear stress at the surface of the fibres (Kelly and Tyson 1965) which then induces stresses within each fibre. In our model, an external tensile stress is applied to the matrix in the direction parallel to the fibre axis so that the mode of deformation is, therefore, axisymmetric. The interfacial stress, τ, generated in this way, induces axial, σ_z, and radial, σ_r, stress components within the fibre which are independent of ϕ.

(b) Derivation of the stress equations

Consider an infinitesimal element of the fibre of length, δz (see figure 1). The element is loaded by the interfacial stress, τ, which acts tangentially to the surface of the fibre. In general, the interfacial stress varies along the length of the fibre. This interfacial stress gives rise to an interfacial force tangential to the surface of the element. This force can be resolved into two components: one parallel to the axis of the fibre given by $\tau.2\pi r.\delta s.\cos\theta$ and the other in the radial direction given by $\tau.2\pi r.\delta s.\sin\theta$ at the fibre surface. Here δs is the length of the surface of the element. The axial stresses generated at z and $z+\delta z$ are represented by $\sigma_z(z)$ and $\sigma_z(z+\delta z)$, respectively. The model is not intended to provide information on the axial stress distribution over the fibre cross-section. The radial stress generated at the surface of the fibre element at z is given by $\sigma_r(z)$.

If the element is at equilibrium, the magnitude of the resultant force on the element in the z direction is zero. This can be expressed as

$$-\sigma_z(z)\pi r^2 + \sigma_z(z+\delta z)\pi[r+\delta r]^2 + \tau 2\pi r\delta s.\cos\theta \approx 0\,. \qquad (2.1)$$

Here δr is the change in r between z and $z + \delta z$. Noting that $\delta s.\cos\theta \approx \delta z$, equation (2.1) can be expanded and simplified to give

$$-\sigma_z(z)r^2 + \sigma_z(z+\delta z)\{r^2 + 2r\delta r + [\delta r]^2\} + 2\tau r\delta z \approx 0\,. \qquad (2.2)$$

We rearrange equation (2.2) to give

$$[\sigma_z(z+\delta z) - \sigma_z(z)]r^2 + 2r\delta r\sigma_z(z+\delta z) + \sigma_z(z+\delta z)[\delta r]^2 + 2\tau r\delta z \approx 0\,. \qquad (2.3)$$

Neglecting terms containing δr with powers higher than unity and dividing throughout by δz gives

$$\frac{\sigma_z(z+\delta z)-\sigma_z(z)}{\delta z} r^2 + 2r\sigma_z(z+\delta z)\frac{\delta r}{\delta z} + 2\tau r \approx 0 . \tag{2.4}$$

In the limit as $\delta z \to 0$

$$r^2 \frac{\mathrm{d}\sigma_z}{\mathrm{d}z} + 2r\sigma_z(z)\frac{\mathrm{d}r}{\mathrm{d}z} + 2\tau r = 0 , \tag{2.5}$$

which can be further simplified to

$$\mathrm{d}[\sigma_z r^2]/\mathrm{d}z + 2\tau r = 0 . \tag{2.6}$$

Equation (2.6) provides a solution to the axial stress distribution in the fibre for any fibre shape if the interfacial stress is known. In the special case of fibres with constant radius, r, equation (2.6) reduces to the form proposed by Kelly and Tyson (1965), i.e.,

$$\mathrm{d}\sigma_z / \mathrm{d}z + 2\tau / r = 0 . \tag{2.7}$$

Similarly, in the radial direction, as the magnitude of the resultant force on the surface of the element is also zero, we can write

$$\sigma_r 2\pi r\delta z - \tau 2\pi r\delta s.\sin\theta \approx 0 . \tag{2.8}$$

Noting that $\delta s.\sin\theta \approx \delta r$, we simplify equation (2.8) to get

$$\sigma_r \delta z - \tau\delta r \approx 0 . \tag{2.9}$$

Divide throughout by δz gives

$$\sigma_r - \tau\delta r / \delta z \approx 0 . \tag{2.10}$$

In the limit as $\delta z \to 0$, we get

$$\sigma_r(z) = \tau \mathrm{d}r / \mathrm{dz} . \tag{2.11}$$

Equation (2.11) gives a value for the radial stress on the surface of the fibre for any fibre shape if the interfacial shear stress is known. Note that the radial stress at any point on the surface of the fibre is proportional to the rate of change of the radius at that point with respect to distance along the axis.

3. Implementing the general model

(a) Definitions and assumptions

The axial and radial stresses along the fibre can be found by solving equations (2.6) and (2.11). The solutions to these equations depend on three factors. These are the fibre shape, the nature of the interfacial shear stress and, for equation (2.6), the boundary condition applied.

The equations were solved for three fibre shapes: a circular cone (figure 2a), a paraboloid of revolution (figure 2b) and an ellipsoid of revolution (figure 2c). A fibre is defined by its axial ratio

$$q = L / r_o , \tag{3.1}$$

where r_o is the radius of the fibre at the origin (Aspden 1994).

[Insert figure 2(a), (b) and (c) here]

For each type of fibre shape, the axial position along the fibre was normalized by dividing the coordinate value on the z-axis by half the fibre length to give a fractional coordinate,

$$Z = z / L , \tag{3.2}$$

in order to ensure that results of stresses can be compared for fibres of different lengths.

In order to obtain an expression for the interfacial shear stress, τ, we use the approach of Kelly and Tyson (1965) who assumed that the matrix surrounding the fibres was rigid and perfectly plastic. Then flow of the matrix along the fibre induces an interfacial shear stress

$$\tau(Z) = \begin{cases} \tau & 0 \le Z \le 1 \\ -\tau & -1 \le Z < 0 \, . \\ 0 & \text{elsewhere} \end{cases} \tag{3.3}$$

This model is reasonable for the many examples of materials where stiff fibres reinforce a weak matrix. It has been estimated that in animal tissues, which are believed to contain pointed fibres (section 1), the matrix yields to shear stresses of only about 0.1 MPa while the fracture stress for the fibres is about 100 MPa (Hukins *et al*. 1984). The model can also be justified at the molecular level since a constant value for $|\tau|$, along the interface, implies a constant number of interactions per unit area between matrix macromolecules and fibre surface throughout the fibre length. Shear of the interface involves overcoming these intermolecular forces at the interface. A constant number of interactions per unit area is to be expected if the macromolecular composition of the matrix and the fibre do not change along the length of the fibre. An alternative approach is to determine an expression for τ as part of the complete solution, by imposing a further condition; when this approach is adopted, the matrix is usually assumed to be elastic (Dow 1963; Rosen 1965).

A boundary condition is required to solve the differential equation (2.6). At the ends of the fibre, the interface ceases to exist so that the stress transfer and, hence, the axial stress are both zero, i.e.

$$\sigma_z (Z = 1) \approx 0 \tag{3.4}$$

(Kelly and Tyson 1965; Agarwal and Broutman 1980).

It is not possible to solve equation (2.6) directly because at the fibre end, Z = 1, the fibre radius r = 0 and there is no solution. In order to obtain a solution, consider the fibre to be truncated by a small amount (as shown in figure 2) so that the fibre half-length, L, is now defined to the truncated end where the radius is r_c. If this is defined as a fraction of the radius at the fibre mid-point, r_o, then

$$R = r_c / r_o \,. \tag{3.5}$$

The equations for the fibre shape then have to be expressed in terms of R and the solutions to equation (2.6) also contain R. The solution for the non-truncated fibre may be found by setting R = 0 in these expressions.

Since the axial stress distribution depends on fibre shape, some shapes are likely to be more effective in reinforcing a composite than others. For effective reinforcement, the total axial stress, Σ_z, given by

$$\Sigma_Z = 2\int_0^1 \sigma_Z(Z)dZ\,, \tag{3.6}$$

should be as high as possible while the peak stress, σ_o , should be as low as possible, in order to avoid fracture. Thus we define the effectiveness, ξ, of a fibre shape by

$$\xi = \Sigma_z / \sigma_o\,. \tag{3.7}$$

The greater the value of ξ, the more effective is the fibre at reinforcing the composite.

This concept can be illustrated by the familiar case of a uniform cylindrical reinforcing fibre. Then equation (2.7) can be solved to give

$$\sigma_z(Z) = 2\tau q(1 - Z) \tag{3.8}$$

which reaches its maximum value at $Z = 0$ with the result that $\sigma_o = 2\tau q$. From equations (3.6) and (3.8) the total stress is given by

$$\Sigma_z = 2\int_0^1 2\tau q(1 - Z)\mathrm{d}Z = 2\tau q\,. \tag{3.9}$$

Dividing Σ_z by σ_o gives the result that $\xi = 1$ for a cylindrical fibre.

(b) Circular conical fibre

The profile of a circular conical fibre is illustrated in figure 2(a). Such a profile can be described by

$$r(Z) = (L / q)[1 - (1 - R)Z]. \quad (3.10)$$

Equation (2.6) was solved to obtain the axial stress distribution within $0 < Z < 1$, subject to the boundary condition of equation (3.4) to give

$$\sigma_z(Z) = \tau q \frac{1}{1-R}\left\{1 - \left(\frac{R}{1-(1-R)Z}\right)^2\right\}. \quad (3.11)$$

To obtain the radial stress distribution along the fibre surface, equation (2.11) was solved to give

$$\sigma_r(Z) = -(\tau / q)[1 - R]. \quad (3.12)$$

In the limiting case as R equals zero, these equations (3.11) and (3.12) become

$$\sigma_z(Z) = \tau q \quad (3.13)$$

and

$$\sigma_r(Z) = -\tau / q, \quad (3.14)$$

respectively.

From equations (3.6) and (3.13), the total axial stress in a conical fibre is given by

$$\Sigma_z = 2\int_0^1 \tau q dZ = 2\tau q, \quad (3.15)$$

as for the cylinder. Equation (3.13) shows that σ_z is constant for a conical fibre so that its maximum value, σ_o, is simply τq. Then, for equation (3.7), the effectiveness of reinforcement is given by $\xi = 2\tau q/\tau q = 2$.

(c) Paraboloidal fibre

The profile of a paraboloidal fibre is illustrated in figure 2(b). Such a profile is described by

$$r(Z) = (L/q)\sqrt{1-(1-R^2)Z}\ . \tag{3.16}$$

Equation (2.6) was solved to obtain the axial stress distribution within $0 < Z < 1$, subject to the boundary condition of equation (3.4) to give

$$\sigma_z(Z) = \tau q \frac{4/3}{1-R^2}\left\{\sqrt{1-(1-R^2)Z} - \frac{R^3}{1-(1-R^2)Z}\right\}. \tag{3.17}$$

To obtain the radial stress distribution along the fibre surface, equation (2.11) was solved to give

$$\sigma_r(Z) = -\frac{\tau}{2q}\left\{\frac{1-R^2}{\sqrt{1-(1-R^2)Z}}\right\}. \tag{3.18}$$

When R equals zero, equations (3.17) and (3.18) become

$$\sigma_z(Z) = \frac{4\tau q}{3}\sqrt{1-Z} \tag{3.19}$$

and

$$\sigma_r(Z) = -\frac{\tau}{2q}\left\{\frac{1}{\sqrt{1-Z}}\right\}, \tag{3.20}$$

respectively.

From equations (3.6) and (3.9), the total axial stress in a paraboloidal fibre is given by

$$\Sigma_z = 2\int_0^1 (4\tau q/3)\sqrt{1-Z}\,\mathrm{d}Z = 16\tau q/9\,. \tag{3.21}$$

According to equation (3.19), σ_z has a maximum value, when $Z = 0$, of $4\tau q/3$. Substituting values of Σ_z and σ_o, for a paraboloidal fibre, into equation (3.7) yields a value for ξ of 4/3.

(d) Ellipsoidal fibre

The profile of an ellipsoidal fibre is illustrated in figure 2(c). Such a profile is described by

$$r(Z) = (L/q)\sqrt{1-(1-R^2)Z^2}\ . \tag{3.22}$$

Equation (2.6) was solved to obtain the axial stress distribution within $0 < Z < 1$, subject to the boundary condition of equation (3.4) to give

$$\sigma_z(Z) = \tau q\left\{\frac{R}{1-(1-R^2)Z^2} + \frac{\sin^{-1}(\sqrt{1-R^2}) - \sin^{-1}(Z\sqrt{1-R^2})}{(\sqrt{1-R^2})[1-(1-R^2)Z^2]} - \frac{Z}{\sqrt{1-(1-R^2)Z^2}}\right\}. \tag{3.23}$$

To obtain the radial stress distribution along the fibre surface, equation (2.11) was solved to give

$$\sigma_r(Z) = -\frac{\tau}{q}\left\{\frac{Z(1-R^2)}{\sqrt{1-(1-R^2)Z^2}}\right\}. \tag{3.24}$$

When *R* equals zero, the equations (3.23) and (3.24) become

$$\sigma_z(Z) = \tau q\left\{\frac{\pi/2 - \sin^{-1} Z}{1-Z^2} - \frac{Z}{\sqrt{1-Z^2}}\right\} \tag{3.25}$$

and

$$\sigma_r(Z) = -\frac{\tau}{q}\left\{\frac{Z}{\sqrt{1-Z^2}}\right\}, \tag{3.26}$$

respectively.

From equations (3.6) and (3.25) the total axial stress in an ellipsoidal fibre is given by

$$\Sigma_z = 2\int_0^1 \tau q\left\{\frac{\pi/2 - \sin^{-1} Z}{1-Z^2} - \frac{Z}{\sqrt{1-Z^2}}\right\}\mathrm{d}Z = 1.66\tau q\ . \tag{3.27}$$

This value was obtained by solving the integral numerically using the trapezoidal rule (Press *et al*. 1992). According to equation (3.25), σ_z has a maximum value of $\pi\tau q/2$ when $Z = 0$. Thus, for an ellipsoidal fibre, $\xi = 1.06$.

4. Predictions

The expressions obtained for axial stresses are multiples of τq (equations 3.13, 3.19 and 3.25) and those for the radial stresses are multiples of τ/q (equations 3.14, 3.20 and 3.26). Using these as normalizing factors for the calculated stresses means that the results may be displayed in a dimensionless form and readily applied to any fibre of known axial ratio for any value of applied shear stress (Aspden 1994).

[Insert figure 3(a), (b) and (c) here]

Figure 3(a)-(c) compares the axial stress distributions for a conical fibre, a paraboloidal fibre and an ellipsoidal fibre. In each case, a comparison with the stress distribution obtained from a uniform cylindrical fibre enables the effect of a taper to be assessed. Note that for plastic deformation of the matrix, the axial stress distribution in a uniform cylindrical fibre never reaches a plateau (Kelly and Tyson 1965; Aspden 1994). Figure 3(a) shows for the conical fibre that the axial stress distribution is uniform. In contrast, the stress distributions from the paraboloidal and ellipsoidal fibre are not; they have axial stress distributions which are intermediate between the two extremes of the cylinder and the cone.

[Insert figure 4(a), (b) and (c) here]

Figure 4(a)-(c) compares the radial stresses for a conical fibre, a paraboloidal fibre and an ellipsoidal fibre. For a conical fibre, the radius, r, decreases linearly with Z, leading to a uniform compressive stress distribution. However, at the end of the fibre $r = 0$, so that there is a discontinuity in the distribution. For the paraboloidal and ellipsoidal fibres, the compressive stress distribution is non-uniform and tends to infinity as r tends to zero.

5. Discussion

An important property of all the tapers considered is to make the axial stress distribution in a fibre more uniform. In a uniform cylindrical fibre, the stress increases linearly from zero, at the ends, to a maximum value at the centre. At the other extreme, the axial stress distribution in a conical fibre is uniform. The intermediate cases of a paraboloidal and an ellipsoidal fibre lead to axial stress distributions which lie between the two extremes, as shown in figures 3 and 4. In tapered fibres there is also a radial (compressive) stress but its magnitude is much smaller than that of the axial stress.

In order to provide effective reinforcement, the total axial stress in a fibre needs to be as high as possible but stress concentrations, approaching the fracture stress, should be avoided. Thus the total stress needs to be as high as possible while the peak stress should be as low as possible. For the simple tapers investigated here, it is then possible to define an effectiveness of reinforcement, ξ, as the total stress divided by the peak stress. According to this criterion, it has been shown that the effectiveness of reinforcement, for the shapes investigated, can be ordered as : cone ($\xi = 2$) > paraboloid ($\xi = 4/3 = 1.33$) > ellipsoid ($\xi = 3.32/\pi = 1.06$) > cylinder ($\xi = 1$).

These results apply to a fibre surrounded by a plastic matrix as defined by Kelly and Tyson (1965). Changing the matrix properties will affect the interfacial stress distribution and so will influence the results. However, equation (2.6) provides a general expression which relates the axial stress, to the fibre radius, r, and interfacial stress, τ. In general, the values of all these parameters may vary along the fibre length. There is also a radial stress, σ_r, which varies along the length of a non-cylindrical fibre and can be calculated from equation (2.11).

Acknowledgements

We thank the University of Aberdeen for an award of a university postgraduate studentship to KLG and the Medical Research Council for an award of a senior fellowship to RMA.

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Figure Captions

Figure 1. Description of a fibre. The fibre axis defines the z-axis of the cylindrical coordinate system whose origin, O, is at the centre of the fibre axis. The interfacial shear stress, $\tau(z)$, acting tangentially to the surface of the fibre, at z, makes an angle, θ, with the z-axis. The internal axial stresses generated at z and $z+\delta z$ are represented by $\sigma_z(z)$ and $\sigma_z(z+\delta z)$, respectively. Here, δz is the length of the fibre element. The curvature length of the fibre element is represented by δs. L, r_o and $r(z)$ represent half the length of a fibre, the radius at the centre of the fibre and the radius of the fibre, at z, respectively.

Figure 2. Profiles of fibre shape for (a) a cone, (b) a paraboloid and (c) an ellipsoid, all with circular cross-sections. Here, r_c represents the truncated radius at the tip of the fibre.

Figure 3. Graphs of dimensionless axial stress versus fractional distance along half a fibre for the following shapes (a) a cone, (b) a paraboloid and (c) an ellipsoid, all with circular cross-sections. These curves are represented by bold lines. The fine lines denote the (dimensionless) stresses obtained for a uniform cylindrical fibre. The true axial stress can be obtained by multiplying the dimensionless axial stress by τq, where τ and q denote a constant interfacial shear stress and the fibre axial ratio, respectively.

Figure 4. Graphs of dimensionless radial stress versus fractional distance along half a fibre for the following shapes (a) a cone, (b) a paraboloid and (c) an ellipsoid, all with circular cross-sections. These curves are represented by bold lines. The negative signs indicate compressive stresses. Not represented on the graphs is the (dimensionless) radial stress for

a uniform cylindrical fibre which is always zero. The true radial stress can be obtained by multiplying the dimensionless radial stress by τ/q, where τ and q denote a constant interfacial shear stress and the fibre axial ratio, respectively.